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ON A TWO-PARAMETER FAMILY OF DISCRETE UNIMODAL RANDOM VARIABLES*

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SUMMARY. This paper studies a new class of **Z**-valued random variables, called beta unimodal, which are 'dot products' (in the sense of Steutel and van Harn, 1979) of the form $U \odot Z$, where U is beta distributed and independent of the **Z**-valued random variable Z. As a particular case, beta unimodality contains the version of α-unimodality on **N**, described in Abouammoh (1987), Steutel (1988). The key result (Theorem 3.8) shows that the space of all beta unimodal probability distributions is isomorphic with the space of all probability measures on **Z**, entailing many similarities between beta unimodality on **Z** and classical unimodality on **R**.

1. Introduction

In order to study discrete analogues of self-decomposability, Steutel and van Harn (1979) (see also van Harn, Steutel, and Vervaat (1982)) introduced a binary operation $\odot$, called dot product, acting on random variables. This product was later used by Steutel (1988) for the characterization of a certain type of generalized discrete unimodality on **N**, called α-unimodality, introduced by Abouammoh (1987, 1988). Denoting the beta distribution with parameters α and ν by $\mathrm{B}(\alpha, \nu)$ and equality in law by $\doteq$, the result can be stated as follows:

Theorem 1.1. (Steutel) *A* **N***-valued random variable* X *is* α*-unimodal (Remark 3.2, 1) if and only if* $X \doteq U \odot Z$*, where* U *is* $\mathrm{B}(\alpha, 1)$*-distributed,* Z *is* **N***-valued, and* U *and* Z *are independent.*

This statement can be viewed as a discrete analogue of a result of Lévy-Shepp (Lévy (1962); Shepp (1962)):
A **R***-valued random variable* X *is unimodal if and only if* $X \doteq UZ$*, where* U *is uniform on* $[0, 1]$*,* Z *is* **R***-valued, and* U *and* Z *are independent.*

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Combining this result with the Representation Theorem of Khinchin and identifying random variables with their probability distributions, one obtains a stronger statement for the continuous case:

Theorem 1.2. (Lévy-Shepp) . *The map $Z \mapsto UZ$ is an affine homeomorphism from the space of all probability measures on* $\mathbf{R}$ *onto the space of all unimodal probability measures on* $\mathbf{R}$.

It is Theorem 1.2 which, using standard techniques from convexity, makes possible an efficient deduction of many results of classical unimodality on $\mathbf{R}$.

Replacing the ordinary product by the dot product, we prove in this paper that an analogue of Theorem 1.2 holds for the discrete case as well, even if we take U to be $\mathrm{B}(\alpha, \nu)$-distributed and Z to be $\mathbf{Z}$-valued (Theorem 3.8). Such a random variable $X \doteq U \odot Z$ is said to be (α, ν)-*unimodal* (at 0) (Definition 3.1) or, in short, *beta unimodal*.

Section 2 gathers technical material on an extended version of dot products, fractional derivatives and differences, and hypergeometric functions, needed for the remaining sections. Section 3 introduces beta unimodality and proves the discrete version (Theorem 3.8) of Theorem 1.2, as well as several other characterizations of discrete beta unimodality. Section 4 exploits the general theory of Section 3 for the comparison of (α, ν)-unimodalities in terms of α and ν, for convolution properties, and for bounds for the variance. Beta unimodality of some standard discrete distributions is also investigated.

2. Preliminaries

We denote by $\boldsymbol{\mathcal{Z}}$ the set of all $\mathbf{Z}$-valued random variables, by $\boldsymbol{\mathcal{P}}(\mathbf{Z})$ the set of all probability measures on $\mathbf{Z}$, and by $\mathrm{C_b}(\mathbf{Z})$ the set of all bounded functions on $\mathbf{Z}$. In general, the associated probability distribution of a random variable $Z \in \boldsymbol{\mathcal{Z}}$ will be denoted by z or (z_n). Similar conventions hold for $\mathbf{N}$. As usual, 1_A stands for the indicator function of a set A. Throughout the remainder of this paper we shall switch freely between random variables and their probability distributions.

Symmetric formulae for measures z on $\mathbf{Z}$ are compressed by the convention $z_0 = z_{+0} + z_{-0}$. The use of this convention is signalled by the notation $\hat{=}$ for equality. So $z_{\pm 0} \hat{=} a_{\pm 0} - a_{\pm 1}$ should be interpreted as shorthand for $z_{+0} = a_{+0} - a_{+1}$, $z_{-0} = a_{-0} - a_{-1}$ and hence $z_0 = a_0 - a_1 - a_{-1}$.

2.1. Spot and dot products The next definition introduces the 'spot product' and extends the notion of dot product (Steutel and van Harn, 1979) to random variables in $\boldsymbol{\mathcal{Z}}$.

Definition 2.1. Let $u \in [0,1]$, $Z \in \boldsymbol{\mathcal{Z}}$, and $f \in \mathrm{C_b}(\mathbf{Z})$.

(1) The *spot product* $u \bullet f$ of u and f is the bounded function defined by:

$$(u \bullet f)(\pm i) = \sum_{n=0}^{i} \binom{i}{n} u^n (1-u)^{i-n} f(\pm n) \quad \text{for } i \geq 0. \qquad \dots (1)$$

(2) The *dot product* $u \odot Z$ of u and Z is the random variable in $\boldsymbol{\mathcal{Z}}$, defined in law by the *dot product* $u \odot z$ of u and the distribution $z = (z_n)$ of Z:

$$(u \odot z)_{\pm i} \hat{=} \sum_{n=i}^{\infty} \binom{n}{i} u^i (1-u)^{n-i} z_{\pm n} \quad \text{for} i \geq 0. \qquad \ldots (2)$$

Equivalently, the dot product can be described by

$$u \odot Z = \sum_{i=1}^{|Z|} \operatorname{sign}(Z) N_i, \qquad \ldots (3)$$

where the N_i are independent Bernoulli variables with success probability u.

Lemma 2.2. *The following hold:*

1) *The map* $u \mapsto u \bullet f$ *from* $[0,1]$ *into* $\mathrm{C_b}(\mathbf{Z})$ *is strictly continuous.*

2) *The maps* $f \mapsto u \bullet f$, $u \in [0,1]$, *from* $\mathrm{C_b}(\mathbf{Z})$ *into itself are equicontinuous with respect to the strict topology.*

3) $(u \odot z)(f) = z(u \bullet f)$ *for any* $z \in \boldsymbol{\mathcal{P}}(\mathbf{Z})$, $f \in \mathrm{C_b}(\mathbf{Z})$, *and* $u \in [0,1]$.

4) *The map* $u \mapsto u \odot z$ *from* $[0,1]$ *into* $\boldsymbol{\mathcal{P}}(\mathbf{Z})$ *is weakly continuous.*

5) *The maps* $z \mapsto u \odot z$, $u \in [0,1]$, *from* $\boldsymbol{\mathcal{P}}(\mathbf{Z})$ *into itself are weakly equicontinuous.*

6) $1 \bullet f = f$, $0 \bullet f = f(0)$, $t \bullet (u \bullet f) = (tu) \bullet f$.

7) $1 \odot Z \doteq Z$, $0 \odot Z \doteq 1$, $t \odot (u \odot Z) \doteq (tu) \odot Z$.

Proof. 1) 2) Since the strict topology coincides with the topology of compact convergence on uniformly bounded sets, these statements follow from the remark that the supnorm of $u \bullet f$ is dominated by the supnorm of f.

3) Indeed,

$$(u \odot z)(f) = \sum_{i=0}^{\infty} f(\pm i)(u \odot z)_{\pm i} = \sum_{i=0}^{\infty} (u \bullet f)(\pm i) z_{\pm i} = z(u \bullet f).$$

4) 5) Apply 3) to 1) and 2).

6) 7) These statements follow from straightforward calculations. □

In continuous unimodality it is convenient to consider the Shepp mapping $Z \mapsto UZ$ of Theorem 1.2 as the transpose of a strictly continuous linear mapping on the space of bounded continuous functions on $\mathbf{R}$. For a discrete analogue, we observe (Lemma 2.2, 3) that the map $u\odot$ is the transpose of the map $u\bullet$. Further, consider the map $U\bullet : \mathrm{C_b}(\mathbf{Z}) \to \mathrm{C_b}(\mathbf{Z})$, defined by compounding:

$$(U \bullet f)(n) = \int_0^1 (u \bullet f)(n) \, d\,\mathrm{B}(u; \alpha, \nu); \qquad \ldots (4)$$

here $\mathrm{B}(\cdot;\alpha,\nu)$ is the beta distribution function with parameters α and ν. By the same procedure we define the map $U\odot : \boldsymbol{\mathcal{P}}(\mathbf{Z}) \to \boldsymbol{\mathcal{P}}(\mathbf{Z})$ by:

$$(U \odot z)_n = \int_0^1 (u \odot z)_n \, d\,\mathrm{B}(u;\alpha,\nu). \qquad \ldots (5)$$

Thus, as in continuous unimodality, we obtain

Lemma 2.3. *The map $U\bullet$ is strictly continuous and $U\odot$ is its transposed map.*

2.2. Fractional derivatives and differences. Let $\nu > 0$, $\nu = m - p$, where m is the least integer strictly greater than ν and $0 < p \leq 1$. Then (Oldham and Spanier, 1974, p.59) for differentiation of arbitrary order ν we have:

$$\frac{d^\nu}{dx^\nu} f(x) = f^{(\nu)}(x) = \frac{d^m}{dx^m} \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} f(t) dt.$$

From Bertin and Theodorescu (1992, Corollary 3.6) we borrow the following result:

Remark 2.4. *Let g be bounded and continuous on $\mathbf{R}$ and let $t > 0$. Then*

$$g(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\nu)} t^{1-\alpha} \left[t^{\nu+\alpha-1} \int_0^1 g(tu) \, d\,\mathrm{B}(u;\alpha,\nu) \right]^{(\nu)}.$$

Further, let α, $\nu > 0$ be fixed. For the description of the inverse of $U\odot$, we introduce a fractional difference type operator $\bigcirc : x \mapsto \bigcirc(x) = (\bigcirc(x)_n)$ on $\boldsymbol{\mathcal{P}}(\mathbf{Z})$, defined by:

$$\bigcirc(x)_{\pm n} \mathrel{\hat{=}} \frac{\Gamma(\alpha)\Gamma(n+\alpha+\nu)}{n!\Gamma(\alpha+\nu)} \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} \frac{(|n|+k)!}{\Gamma(\alpha+|n|+k)} x_{\pm(n+k)}, \qquad \ldots (6)$$

where

$$\binom{\nu}{k} = \frac{\Gamma(\nu+1)}{\Gamma(k+1)\Gamma(\nu-k+1)}.$$

Lemma 2.5. *For any $n \in \mathbf{Z}$, the map $x \mapsto \bigcirc(x)_n$ from the set $\boldsymbol{\mathcal{P}}(\mathbf{Z})$ into $\mathbf{R}$ is affine and weakly continuous.*

Proof. For $n > 0$ we have $\bigcirc(x)_n = x(f) = C \sum_{i=0}^{\infty} x_i f_i$, where C is a constant,

$$f_i = \begin{cases} 0 & \text{for } i < n \\ (-1)^{i-n} \binom{\nu}{i-n} \frac{i!}{\Gamma(\alpha+i)} & \text{for } i \geq n. \end{cases}$$

It is therefore sufficient that f be bounded. This is obvious for $\nu \in \mathbf{N}$ and results from $(i-n)!\pi\Gamma(\alpha+i)|f_i| = \Gamma(\nu+1)i!|\sin \pi\nu|\Gamma(i-n-\nu)$ for $\nu \notin \mathbf{N}$. The proof for $n \leq 0$ is similar. $\square$

2.3. The theorem of Gauss. Let $(a)_r$ be the Pochhammer symbol of $a \in \mathbf{R}$, defined by $(a)_0 = a$ and $(a)_r = a(a+1)\cdots(a+r-1)$ for $r \in \mathbf{N} \setminus \{0\}$. From the Theorem of Gauss (Bailey 1935, p. 2) for hypergeometric functions in the form

$$_2F_1(-k, b, c; 1) = \frac{(c-b)_k}{(c)_k},$$

with $c = \nu - k + 1$ and $b = \nu$ or $b = \alpha + \nu$, we deduce the following equalities:

$$0 = \sum_{j=0}^{k} (-1)^j \frac{\Gamma(\nu + k - j)}{j!(k-j)!\Gamma(\nu - j + 1)} \text{for } k \geq 1. \qquad \ldots (7)$$

$$1 = \sum_{j=0}^{k} (-1)^j \frac{k!\Gamma(\alpha + \nu + k - j)\Gamma(\alpha)\Gamma(\nu + 1)}{j!(k-j)!\Gamma(\alpha + \nu)\Gamma(\nu - j + 1)\Gamma(\alpha + k)}. \qquad \ldots (8)$$

3. Beta unimodality. Basic properties

Consider the random variable $U \odot Z \in \boldsymbol{\mathcal{Z}}$, with distribution $U \odot z$ given by (5), and let us assume that U is independent of Z.

Definition 3.1. Let α, $\nu > 0$. $X \in \boldsymbol{\mathcal{Z}}$ (or its distribution $x = (x_n)$) is said to be *(α, ν)-unimodal (at 0) if and only if*

$$X \doteq U \odot Z, \qquad \ldots (9)$$

where U and $Z \in \boldsymbol{\mathcal{Z}}$ are independent and U is $\mathrm{B}(\alpha, \nu)$*-distributed; in short, $X \in \boldsymbol{\mathcal{B}}(\alpha, \nu)$ (or $x \in \boldsymbol{\mathcal{B}}(\alpha, \nu)$).*

Remark 3.2. (1) Except at the origin, our definition of $(\alpha, 1)$-unimodality coincides with that of discrete α-unimodality given in Abouammoh (1987, 1988) an extension of the commonly used notion of discrete unimodality in the sense of Keilson and Gerber (1971). Discrete α-unimodality on $\mathbf{N}$ is called α-monotonicity in Steutel (1988).

(2) The definitions of Keilson and Gerber, or Abouammoh, cannot lead to results as stated in Theorem 3.8. This is the rationale behind the special role of the origin in our Definition 3.1; see also Remark 3.6.

By Lemma 2.2 we have

Lemma 3.3. *The map $z \mapsto U \odot z$ from $\boldsymbol{\mathcal{P}}(\mathbf{Z})$ into $\boldsymbol{\mathcal{B}}(\alpha, \nu)$ is affine and weakly continuous.*

An immediate consequence of Definition 3.1 and Lemma 2.2, (7) is

Proposition 3.4. *Let $X \doteq U \odot Z \in \boldsymbol{\mathcal{B}}(\alpha, \nu)$ and let T be a random variable, independent of Z and U, with values in* $[0, 1]$. *Then $T \odot X \in \boldsymbol{\mathcal{B}}(\alpha, \nu)$.*

The next result characterizes beta unimodality.

Theorem 3.5. *Let $X \in \boldsymbol{\mathcal{Z}}$ with distribution $x = (x_n)$. The following are equivalent:*
(1) $X \in \boldsymbol{\mathcal{B}}(\alpha, \nu)$.
(2) $\bigcirc(x) \geq 0$.

Proof. (1) $\Longrightarrow$ (2): By virtue of Lemmas 2.5 and 3.3 it suffices to prove (2) for the case $x^n = U \odot \epsilon^n$, where ϵ^n denotes the degenerate probability measure concentrated at $n \in \mathbf{N}$. From (2) and (9) we obtain, for $0 \leq k \leq n$:

$$x^n_{\pm k} = \binom{n}{k} \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)\Gamma(\nu)} \frac{\Gamma(k+\alpha)\Gamma(n-k+\nu)}{\Gamma(n+\alpha+\nu)}. \qquad \ldots (10)$$

Using (6), (10) and (7), we have

$$\begin{aligned}
&\bigcirc(x^n)_{\pm i} \\
&\quad \hat{=} \quad \frac{\Gamma(\alpha)\Gamma(i+\alpha+\nu)}{i!\Gamma(\alpha+\nu)} \sum_{k=0}^{n-i} (-1)^k \binom{\nu}{k} \frac{(i+k)!}{\Gamma(\alpha+i+k)} x_{\pm(i+k)} \\
&\quad \hat{=} \quad \frac{\Gamma(i+\alpha+\nu)}{i!\Gamma(\nu)} \sum_{k=0}^{n-i} (-1)^k \binom{\nu}{k} \frac{n!}{(n-i-k)!} \frac{\Gamma(n-i-k+\nu)}{\Gamma(n+\alpha+\nu)} \\
&\quad \hat{=} \quad 1 \text{if } i = n, = 0 \text{ if } i \neq n.
\end{aligned}$$

(2) $\Longrightarrow$ (1): For $n \geq 0$, let $z_{\pm n} \,\hat{=}\, \bigcirc(x)_{\pm n}$. Using (6) and interchanging summation order, we are led to

$$\begin{aligned}
\sum_{n \in \mathbf{Z}} z_n \quad &\hat{=} \quad \sum_{n \geq 0} \binom{n+\alpha+\nu-1}{n} \Gamma(\alpha) \\
&\qquad \times \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} \frac{(n+k)!}{\Gamma(n+k+\alpha)} (x_{+(n+k)} + x_{-(n+k)}) \\
&\hat{=} \quad \sum_{l=0}^{\infty} (x_{+l} + x_{-l}) \sum_{k=0}^{l} (-1)^k \frac{l!\Gamma(l+\alpha+\nu-k)\Gamma(\alpha)\Gamma(\nu+1)}{k!(l-k)!\Gamma(\alpha+\nu)\Gamma(\nu+1-l)\Gamma(\alpha+l)}.
\end{aligned}$$

From (8) we obtain

$$\sum_{n \in \mathbf{Z}} z_n \,\hat{=}\, \sum_{l=0}^{\infty} (x_{+l} + x_{-l}) = 1,$$

and hence z is the distribution of an integer valued random variable Z.

It remains to prove that $X \doteq U \odot Z$, for which it suffices that $x_{\pm n} \hat{=} (U \odot z)_{\pm n}$ for $n > 0$. Using (9), (10), and again (7) this follows from

$$\begin{aligned}
&\frac{\Gamma(\alpha+\nu)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(\nu)\Gamma(n+1)} \sum_{k=n}^{\infty} \frac{\Gamma(k+1)\Gamma(k-n+\nu)}{\Gamma(k-n+1)\Gamma(k+\alpha+\nu)} z_{\pm k} \\
&= \frac{\Gamma(n+\alpha)}{\Gamma(\nu)\Gamma(n+1)} \sum_{k=n}^{\infty} \frac{\Gamma(k-n+\nu)}{\Gamma(k-n+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\nu}{j} \frac{(k+j)! x_{\pm(k+j)}}{\Gamma(\alpha+k+j)} \\
&= \frac{\Gamma(n+\alpha)}{\Gamma(\nu)\Gamma(n+1)} \sum_{s=n}^{\infty} (-1)^s \frac{x_{\pm s} s!}{\Gamma(\alpha+s)} \sum_{k=n}^{s} (-1)^k \binom{\nu}{s-k} \frac{\Gamma(k-n+\nu)}{\Gamma(k-n+1)} \\
&= x_n. \square
\end{aligned}$$

Remark 3.6. Theorem 4.2 of Dharmadhikari and Jogdeo (1988, p.103) can be read as follows:

Let the renormalized restrictions of $x = (x_n) \in \boldsymbol{\mathcal{P}}(\mathbf{Z})$ to $\mathbf{N}$ and $-\mathbf{N}$ be $(1,1)$-unimodal. Then x is $(1,1)$-unimodal if and only if $x_0 \geq x_{-1} + x_1$.

It follows from Theorem 3.5 that the corresponding statement holds for (α,ν)-unimodality:

Let the renormalized restrictions of $x = (x_n) \in \boldsymbol{\mathcal{P}}(\mathbf{Z})$ to $\mathbf{N}$ and $-\mathbf{N}$ be (α,ν)-unimodal. Then x is (α,ν)-unimodal if and only if $\bigcirc(x)_0 \geq 0$.

Remark 3.7. (1) For $(\alpha,1)$-unimodality, the necessary and sufficient condition of Theorem 3.5 simplifies to

$$x \in \boldsymbol{\mathcal{B}}(\alpha,1) \iff \forall i \geq 0 \ \ (i+\alpha)x_{\pm i} \geq (i+1)x_{\pm(i+1)}. \qquad \ldots (11)$$

(2) Since a probability measure $x \in \boldsymbol{\mathcal{P}}(\mathbf{N})$ is α-unimodal (at 0) in the sense of Abouammoh (1987, 1988) if and only if (11) holds, Theorem 3.5 is an extension of Theorem 1.1 to **Z**-valued random variables and beta unimodality.

(3) It can be shown that $\boldsymbol{\mathcal{B}}(\alpha,1)$ fits into the scheme for good structures of discrete unimodality given in Bertin and Theodorescu (1989), based upon the theory of Khinchin structures.

Since the final equalities of the (1) $\implies$ (2) part of the proof of Theorem 3.5 show that the map $U\odot$ is an injection, we even proved a stronger result, the discrete analogue of Theorem 1.2:

Theorem 3.8. *The map $Z \mapsto U \odot Z$ is an affine homeomorphism from $\boldsymbol{\mathcal{P}}(\mathbf{Z})$ onto $\boldsymbol{\mathcal{B}}(\alpha,\nu)$.*

Corollary 3.9. *The mapping $\bigcirc$ is the inverse of $U\odot$.*

This statement is our discrete analogue of the Representation Theorem of Khinchin in continuous unimodality, which reads, loosely speaking, as follows:
The inverse of the map $Z \mapsto UZ$ of Theorem 1.2 is given by $g \mapsto -xg'(x)$, where g is the probability density function of the unimodal random variable UZ.

As a consequence of Theorem 3.8, $\mathcal{B}(\alpha,\nu)$ inherits the topological and convex behaviour of $\boldsymbol{\mathcal{P}}(\mathbf{Z})$. In particular, $\boldsymbol{\mathcal{B}}(\alpha,\nu)$ is the weakly closed convex hull of its extreme boundary and this extreme boundary is weakly closed and homeomorphic to $\mathbf{Z}$. The extreme elements of $\boldsymbol{\mathcal{B}}(\alpha,\nu)$ are affinely independent probability measures, given by $x^n = (x^n_k)_{0\le k\le n}$ and $x^{-n} = (x^{-n}_{-k})_{0\le k\le n}$, where

$$x^{\pm n}_{\pm k} = \binom{n}{k}\frac{(\alpha)_k(\nu)_{n-k}}{(\alpha+\nu)_n}. \qquad \ldots (12)$$

From Lemma 2.3 we now see that $U\bullet$ is an injection.

We close this section with several alternative characterizations for discrete beta unimodality.

Proposition 3.10.

$$\boldsymbol{\mathcal{B}}(\alpha,\nu) = \{x \in \boldsymbol{\mathcal{P}}(\mathbf{Z}) : f \in \mathrm{C_b}(\mathbf{Z}),\ U\bullet f \ge 0 \Longrightarrow x(f) \ge 0\}.$$

Proof. The map $U\odot z \mapsto (U\odot z)(f) = z(U\bullet f)$ is affine and continuous on $\boldsymbol{\mathcal{B}}(\alpha,\nu)$, and positive on the extreme boundary. On the other hand, for $x \notin \boldsymbol{\mathcal{B}}(\alpha,\nu)$, the second separation theorem produces a function $f \in \mathrm{C_b}(\mathbf{Z})$ and a constant c, such that $x(f) < c \le \inf\{y(f) : y \in \boldsymbol{\mathcal{B}}(\alpha,\nu\}$. Hence $U\bullet(f-c) \ge 0$ but $x(f-c) < 0$. □

Theorem 3.11. *For $X \in \boldsymbol{\mathcal{Z}}$ the following are equivalent:*
(1) $X \in \boldsymbol{\mathcal{B}}(\alpha,\nu)$.
(2) $[t^{\alpha+\nu-1}E(f(t\odot X))]^{(\nu)} \ge 0$ on $(0,1]$ for every bounded positive function f on $\mathbf{Z}$.
(3) The map $t \mapsto [t^{\alpha+\nu-1}E(f(t\odot X))]^{(\nu-1)}$ is isotone on $(0,1]$ for every bounded positive function f on $\mathbf{Z}$.

Proof. (1) $\Longrightarrow$ (2): Let $x = U\odot z$, $z \in \boldsymbol{\mathcal{P}}(\mathbf{Z})$, and let $f \in \mathrm{C_b}(\mathbf{Z})$. Using Lemma 2.2, 3 and 6, (9), and Remark 2.4 we have:

$$\begin{aligned}
\left[t^{\alpha+\nu-1}(t\odot x)(f)\right]^{(\nu)} &= \left[t^{\alpha+\nu-1}\,x(t\bullet f)\right]^{(\nu)}\\
&= \left[t^{\alpha+\nu-1}\int_0^1 (u\odot z)(t\bullet f)\,d\,\mathrm{B}(u;\alpha,\nu)\right]^{(\nu)}\\
&= \left[t^{\alpha+\nu-1}\int_0^1 z((ut)\bullet f)\,d\,\mathrm{B}(u;\alpha,\nu)\right]^{(\nu)}\\
&= \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)}t^{\alpha-1}\,z(t\bullet f).
\end{aligned}$$

(2) $\Longrightarrow$ (1): Let $f \in \mathrm{C_b}(\mathbf{Z})$ be such that $U \bullet f \geq 0$, and hence also $U \bullet (t \bullet f) = t \bullet (U \bullet f) \geq 0$ for $t \in [0,1]$. Again using Remark 2.4, x satisfies the relations

$$\begin{aligned} \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} x(t \bullet f) &= t^{1-\alpha}\left[t^{\alpha+\nu-1}\int_0^1 x((ut)\bullet f)\, d\,\mathrm{B}(u;\alpha,\nu)\right]^{(\nu)} \\ &= t^{1-\alpha}\left[t^{\alpha+\nu-1} x\left(\int_0^1 (ut)\bullet f\, d\,\mathrm{B}(u;\alpha,\nu)\right)\right]^{(\nu)} \\ &= t^{1-\alpha}\left[t^{\alpha+\nu-1} x(U \bullet (t \bullet f))\right]^{(\nu)} \\ &= t^{1-\alpha}\left[t^{\alpha+\nu-1} (t \odot x)(U \bullet f)\right]^{(\nu)} \geq 0. \end{aligned}$$

The result now follows from Proposition 3.10. □

Remark 3.12. (1) For $(\alpha, 1)$-unimodality, the equivalence of Theorem 3.11 is a discrete analogue of the defining property for α-unimodality on $\mathbf{R}$ (Olshen and Savage 1970, Definition 1). For $X \in \boldsymbol{\mathcal{N}}$ and $\nu = 1$ the equivalence was proved in Alzaid and Al-Osh (1990); the case $X \in \boldsymbol{\mathcal{N}}$ and $\nu > 0$ was examined in Kizer (1990).

(2) By virtue of Theorem 3.11, $(1, \nu)$-unimodality may be viewed as a discrete analogue for a type of monotonicity (unimodality) introduced by Pestana (1980).

The next characterization of beta unimodality for $X \in \boldsymbol{\mathcal{N}}$ is formulated in terms of its generating function G_X.

Theorem 3.13. *Let $X \in \boldsymbol{\mathcal{N}}$ with distribution $x = (x_n)$. The following are equivalent:*
(1) $X \in \boldsymbol{\mathcal{B}}(\alpha, \nu)$.
(2) There exists $Z \in \boldsymbol{\mathcal{N}}$ such that

$$G_X(s) = \int_0^1 G_Z(1 - w + ws)\, d\,\mathrm{B}(w;\alpha,\nu) \text{ for any } s \in [0,1]. \qquad \ldots (13)$$

(3) There exists $Z \in \boldsymbol{\mathcal{N}}$ such that

$$\frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} t^{\alpha-1}\, G_Z(1-t) = \frac{d^\nu}{dt^\nu}\left\{t^{\alpha+\nu-1}\, G_X(1-t)\right\},\ t \in (0,1]. \qquad \ldots (14)$$

Proof. (1) $\Longleftrightarrow$ (2): Since the probability generating function G_Y of $Y \doteq u \odot Z$ satisfies

$$G_X(s) = G_Z(1 - u + us), \qquad \ldots (15)$$

this results from Definition 3.1 and the analyticity of the probability generating function in the unit circle.

(2) $\Longleftrightarrow$ (3): This follows from the inversion formula given in Remark 2.4. □

By Remark 3.6, Theorem 3.13 can also be applied to discrete beta unimodality on $\mathbf{Z}$.

4. Further properties of beta unimodality

We begin with some hierarchy properties.

Proposition 4.1. *The following hold:*
(1) $\boldsymbol{\mathcal{B}}(\alpha,\nu) \subset \boldsymbol{\mathcal{B}}(\alpha,\nu_1)$ *for* $0 < \nu_1 \leq \nu$.
(2) $\boldsymbol{\mathcal{B}}(\alpha,\nu) \subset \boldsymbol{\mathcal{B}}(\alpha+\mu,\nu-\mu)$ *for* $0 \leq \mu < \nu$.

Proof. (1): If X is (α,ν)-unimodal, then $X \doteq U \odot Z$, where U is $\mathrm{B}(\alpha,\nu)$-distributed. Let now U_1 be $\mathrm{B}(\alpha,\nu_1)$-distributed, let U_2 be $\mathrm{B}(\alpha+\nu_1,\nu-\nu_1)$-distributed, and assume that these random variables are independent. Since

$$\begin{aligned}
&\int_0^1 y^{\alpha+\nu_1-1}(1-y)^{\nu-\nu_1}\left(\frac{x}{y}\right)^{\alpha-1}\left(1-\frac{x}{y}\right)^{\nu_1-1} dy \\
&= x^{\alpha-1}\int_x^1 (1-y)^{\nu-\nu_1-1}(y-x)^{\nu_1-1}\,dy \\
&\quad (\text{put } y-x=t(1-x)) \\
&= x^{\alpha-1}(1-x)^{\nu-1}\int_0^1 t^{\nu_1-1}(1-t)^{\nu-\nu_1-1}\,dt,
\end{aligned}$$

we have $U = U_1U_2$ and, by Lemma 2.2, (6),

$$X \doteq U \odot Z \doteq U_1 \odot (U_2 \odot Z) \doteq U_1 \odot Z_1.$$

(2): Similarly, one may take $U = U_3U_1$, where U_3 is $\mathrm{B}(\alpha+\mu,\nu-\mu)$-distributed. □

From (12) one concludes that a distribution x is extreme in $\boldsymbol{\mathcal{B}}(\alpha,1)$ if and only if its support is finite, contained in either $\mathbf{N}$ or $-\mathbf{N}$, and (11) holds with equality on the support. Hence, by virtue of Theorem 3.8:

Proposition 4.2. $\boldsymbol{\mathcal{B}}(\alpha,1) \subset \boldsymbol{\mathcal{B}}(\alpha_1,1)$ *for* $0 < \alpha \leq \alpha_1$, *with equality only for* $\alpha = \alpha_1$.

Remark 4.3. Proposition 4.2 extends a result of Steutel (1988, p. 139). It is also a discrete analogue of a result of Olshen and Savage (1970, Theorem 7).

Another useful relation is obtained by combining (12) and Lemma 2.3 :

Proposition 4.4. *Let* $\nu = 1$. *The unique solution of the equation* $U \bullet f = 1_{\{n\}}$ *is given by*

$$\alpha f(\pm n) = \begin{cases} (\alpha+n)1_{\{\pm n\}} - (n+1)1_{\{\pm(n+1)\}} & \text{for } n > 0, \\ \alpha 1_{\{0\}} - 1_{\{1\}} - 1_{\{-1\}} & \text{for } n = 0. \end{cases}$$

As a result, we see that $(\alpha,1)$-unimodality implies, as for continuous unimodality, lower bounds for the variance:

Corollary 4.5. *Let* $X \in \boldsymbol{\mathcal{B}}(\alpha, 1)$. *Then*

$$\alpha(\alpha+2)var(X) - \alpha|\operatorname{E}(X)| - \operatorname{E}(X)^2 \geq 0.$$

Equality holds if and only if X *is extreme.*

Proof. Let Z be a random variable with distribution $z = \bigcirc(x)$. The function $\operatorname{Var}(Z)$ is a, not necessarily finite, positive function on $\boldsymbol{\mathcal{P}}(\mathbf{Z})$, vanishing only on the extreme boundary of $\boldsymbol{\mathcal{P}}(\mathbf{Z})$. By virtue of Theorem 3.8, the function $\operatorname{Var} \circ \bigcirc(x)$ on $\boldsymbol{\mathcal{B}}(\alpha, 1)$ has the same properties. From Proposition 4.4 it follows that $\operatorname{E}(Z) = \alpha/(\alpha+1)\operatorname{E}(X)$ and

$$\operatorname{E}(Z^2) = (\alpha+2)\operatorname{E}(X^2)/\alpha - |\operatorname{E}(X)|/\alpha$$

whenever these quantities exist. Hence

$$\operatorname{var} \circ \bigcirc(x) = (\alpha+2)\operatorname{var}(X)/\alpha - |\operatorname{E}(X)|/\alpha - \operatorname{E}(X)^2/\alpha^2. \ \square$$

Let us now turn to convolution. It is easily seen that even $(\alpha, 1)$-unimodality is not in general preserved under convolution. Take, e.g., the following two $(2, 1)$-unimodal distributions

$$x_0 = 0.50, \quad x_1 = 0.40, \quad x_2 = 0.10,$$

$$y_0 = 0.25, \quad y_1 = 0.40, \quad y_2 = 0.35.$$

Their convolution $t = (t_n)_{0 \leq n \leq 4}$ is

$$t_0 = 0.125, \quad t_1 = 0.300, \quad t_2 = 0.360, \quad t_3 = 0.180, \quad t_4 = 0.035.$$

Since $2t_0 = 0.25 < 0.30 = t_1$, t is not $(2, 1)$-unimodal. The convolution of an $(\alpha, 1)$-unimodal and a $(\beta, 1)$-unimodal distribution is even not always $(\alpha+\beta, 1)$-unimodal. For instance, the convolution $w = (w_n)$ of the extreme distributions x^{-2} and y^1 of $\boldsymbol{\mathcal{B}}(0.1, 1)$ satisfies

$$w_{-1} = 0.08304, \quad w_0 = 0.7950, \quad w_1 = 0.0787.$$

It follows that $0.2 \times (w_0) - (z_{-1} + z_1) < 0$ and hence $w \notin \boldsymbol{\mathcal{B}}(0.2, 1)$.

However we have:

Proposition 4.6. *Let* $x \in \boldsymbol{\mathcal{B}}(\alpha, 1)$ *and* $y \in \boldsymbol{\mathcal{B}}(\beta, 1)$. *Then the convolution* $x * y \in \boldsymbol{\mathcal{B}}(\alpha+\beta, 1)$ *whenever* $\alpha \geq 1$ *and* $\beta \geq 1$.

Proof. Since convolution is separately affine and weakly continuous, it suffices to show that $w = x * y \in \boldsymbol{\mathcal{B}}(\alpha+\beta, 1)$ for extreme probability measures $x = x^{\pm n} \in \boldsymbol{\mathcal{B}}(\alpha, 1)$ and $y = y^m \in \boldsymbol{\mathcal{B}}(\beta, 1)$. We assume that $0 \leq m \leq n$. By virtue of (11), for x

concentrated on **N** our statement follows, without any restriction on α and β, from the relations

$$\begin{aligned}(\alpha+\beta+i)w_i &= (\alpha+\beta+i)\sum_{j=0}^{i} x_j y_{i-j} \\ &= \sum_{j=0}^{i}(\alpha+j+\beta+i-j)x_j y_{i-j} \\ &\geq \sum_{j=0}^{i}(i+1-j)x_j y_{i-j+1} + \sum_{j=0}^{i}(j+1)x_{j+1}y_{i-j} \\ &= (i+1)\sum_{j=0}^{i+1} x_j y_{i+1-j} \\ &= (i+1)w_{i+1}.\end{aligned}$$

For $w = x^{-n} * y^m$ we check (11) for $i = 0$, the most elaborate case. From (12) it follows that

$$\begin{aligned}&(\alpha+1)_n(\beta+1)_n((\alpha+\beta)(w_0) - w_{-1} - w_1) \\ &= \sum_{j=0}^{n}(\alpha+\beta)x_{-j}^{-n}y_j^m - \sum_{j=0}^{n}\left(x_{-j}^{-n}y_{j-1}^m + x_{-j}^{-n}y_{j+1}^m\right) \\ &= \sum_{j=0}^{n}\frac{n!}{j!}\frac{(\alpha)_j m!}{j!}(\beta)_{j-1} \\ &\quad \times\left((\alpha+\beta)(\beta+j-1) - j - \frac{(\beta+j-1)(\beta+j)}{j+1}\right).\end{aligned}$$

The last factor is positive, since $\alpha(\beta+j-1) \geq j$ and $\beta \geq \beta+j)/(j+1)$. □

Remark 4.7. (1) The convolution property for $(\alpha, 1)$-unimodal probability measures on **N** has been proved by Alamasatz (1993) by means of generating functions.

(2) Computer experiments suggest that Proposition 4.6 holds with the weaker condition $\alpha+\beta \geq 2$.

The following result slightly generalizes Alzaid and Al-Osh (1990, Theorem 3).

Proposition 4.8. *Let $\nu = 1$ and assume that $Y^{(1)}, \ldots, Y^{(l+1)} \in \boldsymbol{\mathcal{N}}$ and U are independent random variables and that $Y^{(1)}, \ldots, Y^{(l+1)}$ are distributed as Y. For $l \geq 1$ we have:*

$$Y \doteq U \odot (Y^{(1)} + \cdots + Y^{(l+1)}) \qquad \ldots (16)$$

if and only if

$$G_Y(s) = \left[\frac{1}{1+a(1-s)^{\alpha l}}\right]^{1/l} \qquad \ldots (17)$$

for some $a \geq 0$.

Proof. From Theorem 3.13 we obtain the necessary and sufficient condition

$$\alpha G_Y^{l+1}(s) = -(1-s)G'_Y(s) + \alpha G_Y(s).$$

Putting $G_Y = (1+H)^{-1/l}$, this condition reduces to the differential equation $H = \frac{-(1-s)}{\alpha l} H'$ and hence to the solution (17). □

Note that, for $l = 0$, $Y \doteq 0$ is the only solution of (16).

Corollary 4.9. *For* $\alpha = 1$, $Y \doteq U \odot (Y^{(1)} + Y^{(2)})$ *if and only if* Y *has a geometric distribution, possibly degenerated at* 0.

Example 4.10. Let $\delta > 0$, $p \in (0,1)$, and let X have the distribution

$$x_n = \frac{\Gamma(n+\delta)}{\Gamma(\delta) n!} p^\delta (1-p)^n, \quad n \in \mathbf{N};$$

note that for $\delta \in \mathbf{N} \setminus \{0\}$ we obtain the Negative binomial distribution with parameters δ and p. Put $C = \frac{\Gamma(\alpha)\Gamma(n+\alpha+\nu)}{n!\Gamma(\alpha+\nu)}$; in view of (6) and using the Euler transformation, we have:

$$\begin{aligned}
\bigcirc(x)_n &= \\
&= C \frac{p^\delta (1-p)^n}{\Gamma(\delta)} \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} \frac{\Gamma(n+k+\delta)}{\Gamma(n+k+\alpha)} (1-p)^k \\
&= C \frac{p^\delta (1-p)^n}{\Gamma(\delta)} \frac{\Gamma(n+\delta)}{\Gamma(n+\alpha)} \, {}_2F_1(-\nu, n+\delta; n+\alpha; 1-p) \\
&= C \frac{p^\delta (1-p)^n}{\Gamma(\delta)} \frac{\Gamma(n+\delta)}{\Gamma(n+\alpha)} p^{\alpha+\nu-\delta} \, {}_2F_1(n+\alpha+\nu, \alpha-\delta; n+\alpha; 1-p) \geq 0.
\end{aligned}$$

Thus X is (α, ν)-unimodal for $\alpha \geq \delta$. Notice that X is $(1,1)$-unimodal if and only if $\delta(1-p) \leq 1$.

Example 4.11. Consider the Binomial distribution with parameters m and $p \in (0,1)$. For $m = 1$, the distribution is (α, ν)-unimodal at 0 if and only if $p \leq \alpha/(\alpha+\nu)$. For $\nu = 1$ the same condition is necessary and sufficient for any $m \geq 1$.

Example 4.12. The Poisson distribution $x = (x_n)$ with parameter λ is (α, ν)-unimodal if and only if, for any $n \in \mathbf{N}$,

$$\bigcirc(x)_n = \frac{\Gamma(\alpha)\Gamma(n+\alpha+\nu)}{n!\Gamma(\alpha+\nu)} \frac{e^{-\lambda}\lambda^n}{\Gamma(\alpha+n)} \, {}_1F_1(-\nu; \alpha+n; \lambda) \geq 0.$$

For $\nu = 1$, this is the case if and only if $\lambda \leq \alpha$.

Example 4.13. Consider the random walk on $\mathbf{Z}$ with transition probabilities

$$p_{i,i+1} = p, \quad p_{i,i-1} = q, \quad p_{i,i} = r.$$

Let $y^n \in \boldsymbol{\mathcal{P}}(\mathbf{Z})$ be the n-step probability distribution and assume that y^0 is the probability measure concentrated at 0. Then:

(1) *For* $r \geq \max(p,q)$ *we have* $y^n \in \boldsymbol{\mathcal{B}}(\alpha, 1)$ *for all* n *and all* $\alpha \geq 2$.

(2) *For* $r \geq 1/2$ *the set* $\{n \in \mathbf{Z} : x^n \in \boldsymbol{\mathcal{B}}(1,1)\}$ *is a discrete interval, containing* 0 *and* 1.

Indeed, the normalized restrictions of y^n to $\mathbf{N}$ and to $-\mathbf{N}$ are $(1,1)$-unimodal and hence also $(\alpha,1)$-unimodal if $\alpha \geq 1$ (Proposition 4.2). This is obvious for $n = 0$ and $n = 1$ and follows by induction from the inequalities (for $i \geq 0$)

$$\begin{aligned} y_i^{n+1} &= (ry_i^n + py_{i-1}^n + qy_{i+1}^n) \\ &\geq ry_{i+1}^n + py_i^n + qy_{i+2}^n = y_{i+1}^n. \end{aligned}$$

We conclude the proof of (1) by noting that $\alpha y_0^n \geq 2y_0^n \geq y_{-1}^n + y_{-1}^n$.

Assertion (2) follows by induction from

$$\begin{aligned} y_0^n &= \frac{y_0^{n+1} - qy_1^n - py_{-1}^n}{r} \geq \frac{y_1^{n+1} + y_{-1}^{n+1} - qy_1^n - py_{-1}^n}{r} \\ &= y_1^n + y_{-1}^n + \frac{q(y_0^n + y_2^n - y_1^n) + p(y_0^n + y_{-2}^n - y_{-1}^n)}{r} \geq y_1^n + y_{-1}^n. \end{aligned}$$

Computer experiments suggest that for $\alpha < 2$ no y^n is $(\alpha,1)$-unimodal if n is large enough.

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