COMMUN. STATIST.-THEOR. METH., A10(13), 1345-1354 (1981)

A NOTE ON THE UNIMODALITY OF DISCRETE DISTRIBUTIONS

A. M. Abouammoh and A. F. Mashhour

University of Riyadh
Saudi Arabia

Key Words and Phrases: probability mass function; characteristic function.

ABSTRACT

It is noted that the unimadility property is very important and necessary in many probabilistic-statistical models. In this paper, we consider the definition of discrete unimodality such that the mode may be unique integer or a sequence of consecutive integers. It will be shown that the necessary and sufficient condition for the discrete distribution to be unimodal about a can be given through some canonical representation of its characteristic function (ch.f.). Further, characterization results for some well-known distributions are established.

1. INTRODUCTION

The unimodality property plays a significant role in many important problems of probabilistic-statistical nature. For example, we refer to the role of unimodality in the problem of density estimations, see Wegman (1972) and references therein and in the plausible inference model, see Barndorff-Nielsen (1976). In general, most of the likelihood functions can be shown to be unimodal with respect to the parameters involved.

In spite of the scarcity in the literature of papers dealing with discrete unimodality, we are able to derive some interesting and practicable results. As far as we know Keilson and Gerber (1971), Medgyessy (1972) and Dharmadhikari and Jogdeo (1977) are the only papers so far to consider the discrete unimodality in some detail. In the above mentioned papers, the used technique is heavily dependent on the probability mass function (p.m.f.) where our method is not. Also, most of the results established here are not discussed in those papers.

The distribution P_n, that is, $P_n = \Sigma_{\infty}^{n} p_i$, where p_i is the p.m.f. of P_n, whose support is on the lattice of integers, $n \in I = \{ \ldots,-2,-1,0,1,2,\ldots \}$ is said to be unimodal if there exist at least one integer a such that, $p_n \geq p_{n-1}$ for all $n \leq a$ and $p_{n+1} \leq p_n$ for all $n \geq a$, the point a is called the mode of the distribution P_n. It is noticed that the mode may be one integer or a sequence of consecutive integers. The distribution P_n may consist of a single point mass or it may be bounded from one side both sides or unbounded. It is, sometimes, said that the distribution P_n is unimodal about a, with mode at $n = a$ or with vertex at $n = a$ to mean the same thing and the ch.f. of the discrete distribution P_n is defined by $p(t) = \Sigma_n e^{itn} p_n$.

2. THE MAIN RESULTS

In this section we give three main results that characterize the discrete unimodal distributions. The following theorem characterizes the discrete unimodal distributions through their distribution and probability mass functions.

COMMUN. STATIST.-THEOR. METH., A10(13), 1345-1354 (1981)

A NOTE ON THE UNIMODALITY OF DISCRETE DISTRIBUTIONS

A. M. Abouammoh and A. F. Mashhour

University of Riyadh
Saudi Arabia

Key Words and Phrases: probability mass function; characteristic function.

ABSTRACT

It is noted that the unimadility property is very important and necessary in many probabilistic-statistical models. In this paper, we consider the definition of discrete unimodality such that the mode may be unique integer or a sequence of consecutive integers. It will be shown that the necessary and sufficient condition for the discrete distribution to be unimodal about a can be given through some canonical representation of its characteristic function (ch.f.). Further, characterization results for some well-known distributions are established.

1. INTRODUCTION

The unimodality property plays a significant role in many important problems of probabilistic-statistical nature. For

$$v(t) = p(t) + i\ (1 - e^{it})\ p'(t) \tag{2.6}$$

where $v(t) = \Sigma_n\ e^{itn}\ v_n$ is some ch.f..
Similarly, the second condition of theorem 2.2 leads to

$$w(t) = e^{it}\ p(t) + i(1-e^{it})\ p'(t) \tag{2.7}$$

where $w(t) = \Sigma_n e^{itn}\ w_n$ is some ch.f..

To prove the converse, we need to show that if p(t) is the ch.f. of a discrete distribution P_n, and the functions v(t) and w(t) are ch.fs., then P_n is unimodal about zero. Thus by applying the inversion theorem of the ch.fs. or simply by equating the coefficients of e^{itn} in (2.6) one gets

$$v_n = (1-n)\ p_n + (n-1)\ p_{n-1} \tag{2.8}$$

Also equation (2.7) gives

$$w_{n-1} = (1-n)\ p_{n-1} + (n-1)\ p_{n-2} \tag{2.9}.$$

From equations (2.8), (2.9) and theorem 2.1, it follows directly that P_n is unimodal about zero.

Using the above theorem and corollary 1, one can prove the following

<u>Corollary 2</u>: The discrete distribution P_n whose ch.f. is p(t) is said to be unimodal with vertex n = a, iff the functions

$$v(t) = \{(1+a)-a\ e^{it}\}\ p(t) + \{i(1-e^{it})\}\ p'(t)$$

and

$$w(t) = \{a-(a-1)\ e^{it}\}\ p(t) + \{i(1-e^{it})\}\ p'(t)$$

represent some ch.fs..
For simplification, we replace, from now onwards, $i/(e^{it} - 1)$ by A(t).

In the next theorem, we give other characterization result of unimodal discrete distributions via some canonical representations of their ch.fs..

Theorem 2.3: The discrete distribution P_n is said to be unimodal about zero, iff its ch.f. can be represented as

$$p(t) = A(t)\, e^{it} \int_0^t e^{-iu}\, v(u)\, du$$

and

$$p(t) = A(t) \int_0^t w(u)\, du$$

where v(u) and w(u) are some ch.fs..

Proof: By virtue of theorem 2.2, the distribution P_n is unimodal about zero, iff its ch.f. satisfies the relations

$$p'(t) + A(t)\, p(t) = A(t)\, v(t) \qquad (2.10)$$

and

$$p'(t) + e^{it} A(t)\, p(t) = A(t)\, w(t) \qquad (2.11)$$

where v(t) and w(t) are some ch.fs.. Equation (2.10) represents a first order linear differential equation whose integrating factor is

$$I(t) = \exp\{\int A(t)\, dt\} = 1 - e^{-it}.$$

Thus the solution of the differential equation (2.10) is

$$p(t)\, I(t) - P(0)\, I(0) = \int_0^t I(u)\, A(u)\, v(u)\, du,$$

that is

$$p(t) = A(t)\, e^{it} \int_0^t e^{-iu}\, v(u)\, du$$

where v(u) is some ch.f.. It is clear also that equation (2.11) is a first order linear differential equation whose solution is of the form

$$p(t) = A(t) \int_0^t w(u)\, du.$$

This concludes the proof of the theorem.

From theorem 2.3 and corollary 2, one can establish the following result.

Corollary 3: The n&s condition for the discret distribution P_n to be unimodal about n = a is that its ch.f. p(t) can be represented as

$$p(t) = A(t)\, e^{i(a+1)t} \int_0^t e^{-i(a+1)u}\, v(u)\, du$$

and

$$p(t) = A(t)\, e^{iat} \int_0^t e^{-iau}\, w(u)\, du$$

where v(u) and w(u) are some ch.f..

Remark 1: It can be easily seen from corollary 3, that the relatation between the ch.fs. v(u) and w(u) can be explicitly written as

$$w(t) = v(t) + ie^{i(t+a)} \int_0^t e^{-i(u+a)}\, v(u)\, du \quad (2.12)$$

3. EXAMPLES

It is realized that some well-known discrete distributions in statistics are unimodal and thus can be characterized by finding the corresponding ch.fs. v(.) and w(.) as described in theorems 2.2 and 2.3.

I. The Binomial Distribution: The p.m.f. of the binomial distribution P_n is

$$P_n = {}^mC_n\, p^n\, q^{m-n}, \qquad n = 0,1,2,\dots,m$$

and $P_n = 0$ otherwise, where $p + q = 1$, From the p.m.f. one gets

$$P_n / P_{n-1} = 1 + \{ (m+1)p - n / nq \}$$

that is p_n is non-decreasing in n if $n < (m+1)p$ and is non-increasing in n if $n > (m+1)p$ and if $(m+1)p$ is some integer then $P_n = P_{n-1}$, thus the mode a of P_n is $a \in \{(m+1)p - 1, (m+1)p\}$ if $(m+1)p$ is some integer or $a < (m+1)p$, that is, a is the greatest integer less than $(m+1)p$ if $(m+1)p$ is not integer, see also Feller (1968, p. 151). Therefore, the binomial distribution is unimodal. The ch.f. of the binomial distribution is

$$p(t) = (q + p\, e^{it})^m$$

using this and corollary 2, one can show the following.

Theorem 3.1: The discrete distribution P_n is said to be binomial iff its ch.f. p(t) can have the representations

$$p(t) = A(t)\ e^{i(a+1)t} \int_0^t e^{-i(a+1)u}\ v(u)\ du$$

and

$$p(t) = A(t)\ e^{iat} \int_0^t e^{-iau}\ w(u)\ du,$$

where

$v(u) = e^{-iau}(q+p\ e^{iu})^{m-1}\ \{p\ e^{2iu}(m-a) + e^{iu}(p-mp+ap-aq)+a(a+1)\}$

and $w(u) = v(u) + i\ e^{i(a+u)} \int_0^u e^{-i(a+t)} v(t)\ dt$, and a is the mode of the distribution.

Remark 2: It is not difficult to rewrite theorem 3.1 for the uniform and degenerate distributions, since the degenerate distribution is a special case of the uniform distribution which in turn is a special case of the binomial distribution.

II. The Negative Binomial Distribution: The p.m.f. of the negative binomial distribution is

$$\theta_n = {}^{-r}C_n\ p^r\ (-q)^n,\ n=0,1,2,...$$

and $\theta_n = 0$ otherwise where $p + q = 1$ and $r > 0$. From the p.m.f. of θ_n one gets

$$\theta_n / \theta_{n-1} = 1 + \{q(r-1) - np\} / n .$$

Hence, the negative binomial distribution is unimodal about a where $a \in [\ \{q(r-1)/p\} - 1\ ,\ q(r-1)/p\ ]$ if $q(r-1)/p$ is some integer or $a \leq q(r-1)/p$, that is the smallest integer less than or equal $q(r-1)/p$ if $q(r-1)/p$ is not integer. Further the ch.f. $\theta(t)$ of the negative binomial distribution is

$$\theta(t) = \{\ p/(1 - q\ e^{it})\ \}^r$$

Using the above and corollary 2, the following theorem can be established.

Theorem 3.2: The discrete distribution θ_n is said to be negative binomial iff its ch.f. can be represented by

$$\theta(t) = A(t)\ e^{i(a+1)t} \int_0^t e^{-i(a+1)u}\ v(u)\ du$$

and

$$\theta(t) = A(t)\, e^{iat} \int_0^t e^{-iau}\, w(u)\, du$$

where

$v(u) = e^{-iau} \{p\,/(1-q\, e^{iu})\}^r \,[\{epq\, e^{iu}\,/p(1-q\, e^{iu})\} - a(e^{iu}-1)+ 1]$,

$w(u) = v(u) + ie^{i(u+a)} \int_0^u e^{-i(t+a)} v(t)\, dt$, and a is the mode of the distribution.

Remark 3: The above theorem can be applied to the geometric distribution which is a special of the negative binomial.

III. The Poisson Distribution: The p.m.f. of the Poisson distribution P_n is

$$p_n = e^{-\lambda}\ \lambda^m/\, n!, \qquad n = 1,2,...$$

and zero otherwise. We notice that from

$$p_{n+1}\, /\, p_n = \lambda\, /\, (n+1)$$

which shows that the Poisson distribution is unimodal and the mode depends on the value of λ. In particular, if $\lambda = m$, then a = { m-1, m }, m = 1,2,3,... where a is the mode of the distribution and if $m - 1 < \lambda < m$, then $a = m - 1$. Thus the Poisson distribution is unimodal about $a \neq 0$ for all values of $\lambda \geq 1$ and is unimodal about zero for $\lambda < 1$. The ch.f. of the Poisson distribution is $p(t) = \exp\{ - \lambda(1 - e^{it})\}$.

Thus the following theorem can be proved by using the above results and corollary 2.

Theorem 3.3: The discrete distribution P_n is said to be Poisson, iff its ch.f. is represented in the form

$$p(t) = A(t)\, e^{i(a+1)t} \int_0^t e^{-i(a+1)u}\, v(u)\, du$$

and

$$p(t) = A(t)\, e^{iat} \int_0^t e^{-iau}\, v(u)\, du,$$

where

$v(u) = \exp[- \{iau + \lambda\ (1 - e^{iu})\}]\ \ .\ [(e^{iu} - 1)(e^{iu} - a) + 1]$,

and $w(u) = v(u) + i\, e^{i(u+a)} \int_0^u e^{-i(t+a)}\, v(t)\, dt$, the mode a may take the values 0,1,2,... or the pair of values (0,1), (1,2), (2,3),... .

The results of section 2 of this paper can be extended for the multimodal discrete distribution case by, roughly, deviding the multimodal distribution to n unimodal functions, see Abouammoh (1980) where such technique is applied to to continuous distributions.

BIBLIOGRAPHY

Abouammoh, A. M.,(1980). On multimodal statistical distributions. To appear in The Bull. Fac. Sci., Riyadh, 13.

Barndorff-Nielsen, O.,(1976). Plausibility inference. J. Roy. Statist. Soc. Ser. B, 38, 103-131.

Dharmadhikari, S. W. & Jogdeo, K.,(1976). On characterization of unimodality of discrete distributions. Ann. Inst. Statist. Math., 28, 9-18.

Feller, W., (1968). An Introduction to Probability Theory and its Applications. Vol. 1, Wiley.

Keilson, J. & Gerber, B., (1971). Some results for discrete unimodality. J. Amer. Statist. Assoc., 66, 368-389.

Medgyessy, P.,(1972). On the unimodality of discrete distributions. Periodica Mathematica Hungarica, 2, 245-257.

Wegman, E. J.,(1972). Nonparametric probability density estimation I: A summary of availabe methods. Technometrics, 14, 533-546.

Received October, 1980. Retyped December, 1980.

Recommended by E. J. Wegman, Office of Naval Research, Arlington, VA.