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Abstract: In most practical situations the conditional mean remaining life of a device depends on its expired lifetime. We introduce new criteria of aging in terms of the conditional mean remaining life. The implications among these criteria and some other existing concepts of aging are developed. The k-order classes of the mean remaining life criteria are discussed.

AMS 1980 Subject Classifications: Primary 60K10; Secondary 62N05.

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1. Introduction

In this paper we consider aging as the concept that a new system has a longer remaining life than an old one. Broadly speaking there are two approaches for classifying the aging property, namely, the failure rate (FR) and the mean remaining life (MRL). It is noted that (from their corresponding definitions) the MRL depends for its behaviour on large values of the underlying random variable whereas the FR and its other generalized classes "failure rate average (FRA)" and "new better (worse) than used NBU (NWU)" are more sensitive to small values of the (lifetime) random variable. In literature, interest is mainly given to the increasing (decreasing) failure rate IFR (DFR), IFRA, (DFRA), etc. Some of these classes of life distributions arise in maintenance and replacement policies. The MRL class is also receiving a good deal of current attention, especially in biometry, actuarial studies and reliability engineering; see Joe and Proschan (1984) and Guess, Hollander and Proschan (1986). Also the MRL function determines the corresponding distribution uniquely; see Gupta (1981).

Let T be a random variable of system life time. The survival function is defined by

$$\bar{F}(t) = 1 - F(t)$$

where $F(t)$ is the distribution function. Assume that the system is functioning at $t = 0$, so that $\bar{F}(0) = 1$. Let $F(t)$ be differentiable, that is

$$f(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\bar{F}(t)$$

is the density function of T. The failure rate $r(t)$ is defined by

$$r(t) = \lim_{\delta t \to 0} \frac{1}{\delta t} P(T \leqslant t + \delta t \mid T > t) = f(t)/\bar{F}(t).$$

The main concepts of aging in statistical literature are given in Definitions 1.1 and 1.2 below. Theorem 1.3 presents a limiting approach for some implications between classes of life distributions. The new criteria for the MRL are introduced in Section 2. The implications between these criteria and the existing criteria are also investigated. In Section 3 we introduce the k-order MRL criteria in parallel to that for k-HNBUE introduced by Basu and Ebrahimi (1984); see also Basu and

Ebrahimi (1985) and Klefsjö (1985). We shall use the terms increasing and decreasing for non-decreasing and non-increasing, respectively. Next we give some basic definitions.

Definition 1.1. The distribution function $F(t)$, its survival functions $\bar{F}(t)$ or its corresponding random variable T is said to have the following properties.

(i) IFR (DFR) if $r(t)$ is increasing (decreasing) in $t > 0$.

(ii) IFRA (DFRA) if $t^{-1}\int_0^t r(x)\,dx$ is increasing (decreasing) in $t > 0$.

(iii) NBU (NWU) if $\bar{F}(t+s) \leqslant (\geqslant)\ \bar{F}(t)\bar{F}(s)$, for all s, $t > 0$.

(iv) new better (worse) than used in expectation NBUE (NWUE) if

$$\int_t^\infty \bar{F}(x)\,dx \leqslant (\geqslant)\ \mu\bar{F}(t) \quad \text{for } t > 0,$$

where $\mu = \int_0^\infty \bar{F}(x)\,dx < \infty$.

(v) decreasing (increasing) mean remaining life DMRL (IMRL) if $\int_t^\infty \bar{F}(x)\,dx/\bar{F}(t)$ is decreasing (increasing) in $t > 0$.

(vi) harmonic new better (worse) than used in expectation HNBUE (HNWUE) if

$$\int_t^\infty \bar{F}(x)\,dx \leqslant (\geqslant)\ \mu \exp(-t/\mu) \quad \text{for } t > 0.$$

For implications and properties of (i)–(v), see Bryson and Siddiqui (1969), and Barlow and Proschan (1981). Property (vi) was introduced by Rolski (1975).

Following the same line, Loh (1984) and Abouammoh and Ahmed (1988) introduced two other concepts of aging.

Definition 1.2. Let T be a life time random variables as in Definition 2.1, then T is said to have the following properties.

(i) new better (worse) than average in failure rate NBAFR (NWAFR) if $L = \lim_{s\to 0} s^{-1} \log \bar{F}(s)$ exists and $t^{-1}\log \bar{F}(t) \leqslant L$, $t > 0$.

(ii) new better (worse) than used in failure rate NBUFR (NWUFR) if L exists and

$$\frac{d}{dt}\log \bar{F}(t) \leqslant L, \quad t > 0.$$

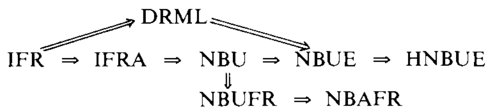

Fig. 1.

We summarize the implications between these classes in Fig. 1.

It should be emphasized that some of these implications can be viewed as limiting forms as follows:

Theorem 1.3. *Let $F(t)$ be the distribution function of the non-negative random variable T. Then*

(i) F is NBUFR if it is IFR and

$$\lim_{t\to 0} r(t)$$

exists.

(ii) F is NBAFR if it is IFRA and

$$\lim_{t\to 0} t^{-1}\int_0^\infty r(x)\,dx$$

exists.

(iii) F is NBUE if it is DMRL and

$$\lim_{t-\to 0}\int_t^\infty \bar{F}(x)\,dx/\bar{F}(t)$$

exists.

The proof is straightforward and therefore omitted.

2. Mean remaining life

Now, we introduce the following classes of life distributions that express some criteria of aging in terms of the mean remaining life.

Definition 2.1. Let T be a nonnegative random variable with survival function $\bar{F}(t)$, then

(i) $\bar{F}$ is said to have DMRL if

$$\mu(t) = \left[\bar{F}(t)\right]^{-1}\int_t^\infty \bar{F}(x)\,dx$$

is decreasing in $t > 0$.

(ii) $\bar{F}$ is said to have specific interval decreasing mean remaining life average (SIDMRLA) if

$$D(t, s) = t^{-1}\int_s^{s+t}\mu(x)\,\mathrm{d}x$$

is decreasing in t for all s, $t \geqslant 0$.

(iii) $\bar{F}$ is said to have decreasing mean remaining life average (DMRLA) if

$$t^{-1}\int_0^t \mu(y)\,\mathrm{d}y$$

is decreasing in $t \geqslant 0$, i.e.,

$$D(t, 0) = t^{-1}\int_0^t [\bar{F}(y)]^{-1}\int_y^{\infty}\bar{F}(x)\,\mathrm{d}x\,\mathrm{d}y$$

is decreasing in $t > 0$.

(v) $\bar{F}$ is said to have new better than average mean remaining life (NBAMRL) property if

$$\mu(0) \geqslant t^{-1}\int_0^t \mu(y)\,\mathrm{d}y,$$

i.e.,

$$t^{-1}\int_0^t [\bar{F}(y)]^{-1}\int_y^{\infty}\bar{F}(x)\,\mathrm{d}x\,\mathrm{d}y \leqslant \mu,$$

where $\mu = \int_0^\infty \bar{F}(x)\,\mathrm{d}x$.

(v) $\bar{F}$ is said to belong to the class of new better than used mean remaining life (NBUMRL) if $\mu(t) \leqslant \mu$, that is

$$[\bar{F}(t)]^{-1}\int_s^{\infty}\bar{F}(x)\,\mathrm{d}x \leqslant \mu.$$

(vi) $\bar{F}$ is said to belong to the class of decreasing harmonic mean remaining life average (DHMRLA) if

$$t^{-1}\int_0^t \mu^{-1}(x)\,\mathrm{d}x$$

is increasing in $t > 0$, i.e.,

$$\left[\mu^{-1}\int_t^{\infty}\bar{F}(x)\right]^{1/t}\,\mathrm{d}t$$

is decreasing in $t > 0$.

(vii) $\bar{F}$ is said to have new better than used harmonic mean remaining life property (NBUHMRL) if

$$t^{-1}\int_0^t \mu^{-1}(x)\,\mathrm{d}x \leqslant \mu^{-1},$$

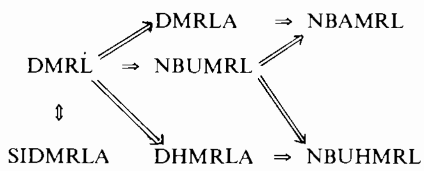

Fig. 2.

i.e.,

$$\mu^{-1}\int_t^{\infty}\bar{F}(x)\,\mathrm{d}x \leqslant \exp(-t/\mu), \quad t > 0.$$

Note that one can define the corresponding dual classes by reversing the inequality sign or the monotonicity direction, as appropriate in Definition 2.1. This gives the classes of IMRL, SI-IMRLA, IMRLA, NWAMRL, NWUHMRL, IHMRLA and NWUHMRL properties, respectively.

In the rest of this section the implications shown in Fig. 2 are proved.

For simplicity, assume that $\mu(t)$ is integrable and has no more than finitely many discontinuity points in any finite interval.

Theorem 2.1. *SIDMRLA and DMRL are equivalent.*

Proof. It is obvious that if $\mu(t)$ is monotone, then so is $D(s, t)$. Further, $\mu(t)$ is monotone decreasing equivalent to the following relation:

$$\mu(s) \geqslant D(s, t) \geqslant \mu(s+t) \quad \text{for } s \geqslant 0,\ t \geqslant 0. \quad \text{(a)}$$

Then, for $t_2 \geqslant t_1 \geqslant 0$ and $s \geqslant 0$,

$$\begin{aligned} D(t_2, s) - D(t_1, s) &= t_2^{-2}\int_s^{s+t_2}\mu(x)\,\mathrm{d}x - t_1^{-1}\int_s^{s+t_1}\mu(x)\,\mathrm{d}x \\ &= t_2^{-1}(t_1 - t_2)\left[(t_2 - t_2)^{-1}\int_{s+t_1}^{s+t_2}\mu(x)\,\mathrm{d}x - t_1^{-1}\int_s^{s+t_1}\mu(x)\,\mathrm{d}x\right] \\ &= t_2(t_2 - t_1)[\mu(s+t_1) - \mu(s+t_1)]. \end{aligned}$$

Using the earlier relation (a) this implies

$$D(t_2, s) \leqslant D(t_1, s).$$

To prove the converse let $D(t_2, s) \leqslant D(t_1, s)$. Then, taking the limit of both sides as $t_1 \to 0$ gives

$$t_2^{-1}\int_s^{s+t_2}\mu(x)\,\mathrm{d}x \leqslant \mu(s+)$$

for any $t_2 > 0$. Using the same argument for relation (a) implies that $\mu(t)$ is decreasing. This completes the proof.

Theorem 2.2. *DMRL implies DMRLA.*

Proof. Using Theorem 2.1 and Definition 2.1(iii) and setting $s = 0$ gives the required result.

Theorem 2.3. *DMRLA implies NBAMRL.*

Proof. By Definition 2.1(iii), $D(t, 0)$ is decreasing in t, taking the limit of $D(t, 0)$ as t tends to zero and using L'Hospitale's rules, we have

$$\lim_{t\to 0} D(t, 0) = \mu.$$

Since $D(t, 0)$ is decreasing, then $D(t, 0) \leqslant \mu$.

Theorem 2.4. *DMRL implies NBUMRL.*

Proof. By Definition 2.1(i), $\mu(t)$ is decreasing.

$$\lim_{t\to 0}\mu(t) = \mu,$$

thus

$$\mu(t) \leqslant \lim_{t\to 0}\mu(t)$$

and hence the theorem.

Theorem 2.5. *NBUMRL implies NBAMRL.*

Proof. From Definition 2.1(v), we have

$$\mu(t) \leqslant \mu \quad \text{for all } x > 0,$$

integrating both sides upto t gives

$$t^{-1}\int_0^t \mu(x)\,\mathrm{d}x \leqslant \mu.$$

Hence the theorem.

Theorem 2.6. *DMRL implies DHMRLA.*

Proof. DMRL means that

$\mu^{-1}(y)$ is increasing for all $y > 0$,

that is

$t^{-1}\int_0^t \mu^{-1}(y)\,\mathrm{d}y$ is increasing for all $t > 0$.

This implies that

$$t^{-1}\log\left(\mu^{-1}\int_t^\infty F(x)\,\mathrm{d}x\right)$$

is decreasing for all $t > 0$.

Theorem 2.7. *DHMRLA implies NBUHMRL.*

Proof. Definition 2.1(vi) of DHMRLA implies that

$t^{-1}\int_0^t \mu^{-1}(y)\,\mathrm{d}y$ is increasing in $t > 0$.

Taking the limit of this quantity and using L'Hospital's rule gives that

$$t^{-1}\int_0^\infty \mu^{-1}(y)\,\mathrm{d}y \geqslant 1/\mu, \quad \text{for all } t > 0.$$

The last inequality is equivalent to the following

$$\mu^{-1}\int_t^\infty F(x)\,\mathrm{d}x \leqslant \exp(-\mu^{-1}t) \quad \text{for all } t > 0.$$

This proves the theorem.

Theorem 2.8. *NBUMRL implies NBUHRML.*

Proof. The definition of NBUMRL states that $\mu(t) \leqslant \mu$ for all $t \geqslant 0$, i.e.,

$$[\mu(t)]^{-1} \geqslant \mu^{-1} \quad \text{for all } t \geqslant 0.$$

Integrating both sides on a (finite) interval $[0, t]$ gives Definition 2.1(iii).

The difference between the three paths of implications that relates DMRL with NBAMRL and NBUHMRL is of special interest. Examining the behaviour of DMRLA, NBUMRL and DHMRLA it can be seen that DMRLA is more sensitive to the behaviour of the survival function for small values of t, NBUMRL depends on its behaviour for large values of t, whereas DHMRLA depends on its behaviour for small values t. Similarly, NBAMRL and NBUHMRL depend on the behaviour of $\mu(t)$ and $\bar{F}(t)$ for small and large values of t. Thus, the absence of any implications between DMRLA, NBUMRL and DHMRLA or NBAMRL and NBUHMRL is not surprising.

The implications between various classes of life distributions presented earlier can be simplified as in Fig. 3.

Using the fact that the mean remaining life $\mu(t)$ determines the corresponding distribution function $F(t)$ uniquely, where

$$F(t) = 1 - [\mu(0)/\mu(t)]\exp\left[\int_0^t \mu^{-1}(x)\, \mathrm{d}x\right],$$

one can construct distribution functions of these remaining life criteria.

Now consider a distribution function $F(t)$ whose mean remaining life is given by

$$\mu(t) = \begin{cases} \exp(2-t), & 0 \leqslant t \leqslant 1, \\ 3.8 - t, & 1 < t \leqslant 3.8. \end{cases}$$

It is clear that $\mu(0) \geqslant \mu(t)$ for all $t > 0$, whereas $\mu(1) < \mu(1+)$. This means that F has the NBUMRL property but it is not DMRL distribution. Similarly one can construct examples to show that the inverse direction of the implications between these mean remaining criteria are not possible.

Now we present the following characterization result for DMRL, DMRLA and DHMRLA distributions, using the fact that the derivative of a decreasing function is non-positive.

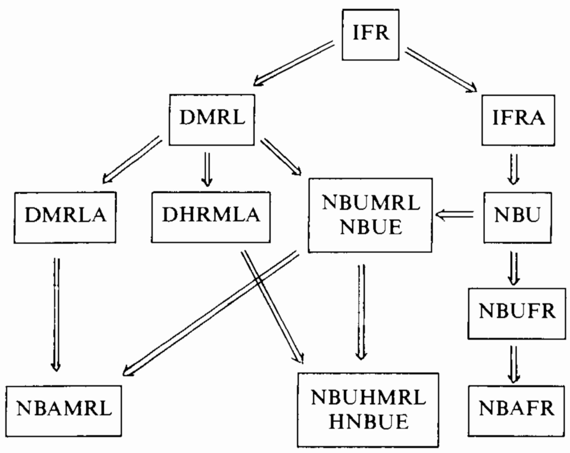

Fig. 3.

Theorem 2.9. *Let $F(t)$ be life distribution and $\mu(t)$ denote the mean remaining life. Then*

(i) *F has the DMRL property iff*

$$\mu(t) \leqslant r^{-1}(t) \quad \text{for } t > 0,$$

(ii) *F has the DMRLA property iff*

$$\mu(t) \leqslant t^{-1}\int_0^t \mu(x)\, \mathrm{d}x \quad \text{for } t > 0,$$

and

(iii) *F has the DHMRLA property iff*

$$\mu(t) \leqslant \log\left[\mu^{-1}\int_t^\infty F(x)\, \mathrm{d}x\right]^{-1} \quad \text{for } t > 0.$$

3. The *k*-order classes

Let F be a life distribution with survival function $\bar{F}(t) = 1 - F(t)$, $t \geqslant 0$, and finite mean μ; then F belongs to the class of k-order HNBUE or simply k-HNBUE if

$$t^{-1}\int_0^\infty \mu^{-k}(x)\, \mathrm{d}x \geqslant \mu^{-k}, \quad t \geqslant 0,$$

where $\mu(x) = [F(x)]\int_x^\infty F(y)\, \mathrm{d}y$. The k-HNWUE is defined similarly.

Basu and Ebrahimi (1985) have proved the following.

Theorem 3.1. *K-HNBUE implies $(k+1)$-HNBUE and $(k+1)$-HNWUE implies k-HNWUE for any $k \geqslant 1$.*

Now we introduce the following k-order classes.

Definition 3.2. Let F be a life distribution with finite mean and

$$\mu(t) = [F(t)]^{-1}\int_t^\infty F(x)\, \mathrm{d}x.$$

(i) F is k-DMRL if

$$[\mu(t)]^k \quad \text{is decreasing in } t \geqslant 0.$$

(ii) F is k-DMRLA if

$$t^{-1}\int_0^t [\mu(x)]^k\, \mathrm{d}x \quad \text{is decreasing in } t > 0,$$

(iii) F is k-NBAMRL if

$$t^{-1}\int_0^t [\mu(x)]^k \,\mathrm{d}x \leqslant \mu^k, \quad t > 0,$$

(iv) F is k-NBUHMRL if

$$[\mu(t)]^k \leqslant \mu^k, \quad t \geqslant 0,$$

and

(v) F is k-DHMRLA if

$$t^{-1}\int_0^t [\mu(x)]^k \,\mathrm{d}x \quad \text{is increasing in } t \geqslant 0.$$

Other classes of distribution that are not represented in terms of MRL can have the k-order forms as follows:

Definition 3.3.

(i) F is k-IFR if

$$[r(t)]^k \quad \text{is increasing in } t \geqslant 0.$$

(ii) F is k-IFRA if

$$t^{-1}\int_0^t [r(x)]^k \,\mathrm{d}x \quad \text{is increasing in } t \geqslant 0.$$

(iii) F is k-NBUFR if

$$[r(0)]^k \leqslant [r(t)]^k \quad \text{for } t \geqslant 0.$$

(iv) F is k-NBAFR if

$$[r(0)]^k \leqslant t^{-1}\int_0^t [r(x)]^k \,\mathrm{d}x \quad \text{for } t \geqslant 0.$$

Note that the dual of the classes, in Definition 3.2 and 3.3 can be defined by either reversing the monotonicity or the inequality directions. For $k = 1$, all the k-order classes are reduced to corresponding classes. Some of these are trivial, that is the k-order and the $(k+1)$-order are equivalent. The one-way implications between the k-order and the $(k+1)$-order are summerized in the following:

Theorem 3.4. *Let F be a life distribution with finite mean μ. Let $\mu(t)$ or $r(t)$ (if needed) be integrable in a finite interval. Then, for distinct values of $k \geqslant 1$,*

(i) *k-NBAMRL implies $(k+1)$-NBAMRL,*
(ii) *$(k+1)$-NWMRL implies k-NWMRL,*
(iii) *k-NBAFR implies $(k+1)$-NBAFR, and*
(iv) *$(k+1)$-NWAFR implies k-NWAFR.*

Proof. All parts (i)–(iv) can be obtained by using Holder's inequality.

Remark 3.5. Insisting on equality and taking all monotone functions to be constant in their corresponding arguments will reduce all classes life distributions in this paper to the exponential distribution.

Remark 3.6. Some properties and the effect of truncation on the MRL function has been studied by Newby (1986). Such study would have corresponding features for the criteria presented in this paper.

Remark 3.7. Abouammoh and Khalique (1987) have considered some test statistics for these classes of life distributions based on the scaled total time on test transform. The closure of these classes under formation of coherent systems, convolutions and mixtures have not yet been studied.

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