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On Characterizations of Discrete Unimodality: A Survey

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The unimodality property is considered as an excellent smoothness characteristic in many problems of statistical inference. The unimodality of continuous random variables has been almost thoroughly investigated by many authors. There are many interesting and practical results, in this direction, available in the literature. In contrast to the continuous case, there are relatively few papers scattered in various research journals dealing with discrete unimodality that is the unimodality of discrete random variables. In this paper, we consider the problem of getting a discrete analogue of some results to continuous unimodality. We present a unified survey of most of thé previous results on such problem. Also, some new results are included.

1. INTRODUCTION

A distribution function (d.f.) $F(x)$ of a continuous random variable (r.v.)X is said to be unimodal if its derivative $F'(x)$ exists everywhere and has a unique finite maximum. A more general definition adopted here, due to Khintchine (1938), states that a d.f. $F(x)$ is unimodal if there exists at least one value $x = a$ such that $F(x)$ is convex for $x \leqslant a$ and concave for $x \geqslant a$. It can be shown that the d.fs. normal, Cauchy and uniform are all unimodal. The d.f. $F(x)$ is strongly unimodal (Ibragimov (1956) and Hajek and Sidak (1967)) if the convolution of F with any unimodal d.f. is unimodal. A nondegenrate d.f. F is strongly unimodal if and only if that the distribution p_n is strongly unimodal if its p.m.f. p_n, $n \in I$ is log-concave.

It is noted that this review is not exhaustive or covers all results related to unimodality in the univariate case. The review does not include some other proposed concepts of unimodality such as totally unimodal, see Pestana (1978).

2. CHARACTERIZATIONS

A very interesting characterization result due to Khintchine for continuous unimodality (see Gnedenko and Kolmogrov (1954)) states that: a continuous d.f. F(x) is unimodal with vertex x = 0 iff its characteristic function (ch.f.) f(t) can be represented as

$$f(t) = \frac{1}{t} \int_0^t g(u)du, \qquad (-\infty < t < \infty) \qquad (2.1)$$

where g(u) is ch.f. .

Such result depends mainly on the fact that F(x) is a unimodal d.f. about x = 0 iff there exists some d.f. G(x) such that $G(x) = F(x) - xF'(x)$. Similar characterizations for discrete unimodality have been tried by Medgyessy (1972), Abouammoh and Mashhour (1981) and Bertin and Theodorcscu (1980).

Theorem 2.1: (Medgyessy (1972)): A discrete distribution p_n, $n \in I$ is unimodal about n_0 iff its p.m.f. p_n satisfies the relation

$$(n_0 + \theta - n)(p_n - p_{n-1}) = q_n \qquad (2.2)$$

ch.f. p(t) satisfies the relation

$$p(t) = A(t)e^{i(n_o+\theta)t} \int_0^t q(u)e^{-i(n_o+\theta)u} du \tag{2.4}$$

where $0 < \theta < 1$, $q(u) = \sum_{n=-\infty}^{\infty} q_n e^{itn}$, $q_{n_o+1} > 0$ and $A(t)=i/(e^{it}-1)$

Theorem 2.5: A necessary and sufficient (n&s) condition for a discrete distribution p_n, $n\varepsilon I$ to be unimodal about $n=n_o$ is that its ch.f. p(t) can be represented in the forms,

$$p(t) = A(t)e^{i(n_o+1)t} \int_0^t e^{-i(n_o+1)u} v(u)du \tag{2.5a}$$

and

$$P(t) = A(t) e^{in_o t} \int_0^t e^{-in_o u} w(u)du \tag{2.5b}$$

where $v(u) = \sum_{n=-\infty}^{\infty} v_n e^{iun}$, $w(u) = \sum_{n=-\infty}^{\infty} w_n e^{iun}$

and $A(t) = i/(e^{it} - 1)$.

Also, Theorem 2.5 can be expressed in a simple form, i.e. in terms of one condition, as follows

Theorem 2.6: A n&s condition for a discrete distribution p_n, $n\varepsilon I$ to be unimodal about n=0 is that its ch.f. p(t) has a finite second derivative and it satisfies.

$$p''(t) + i[1+2B(t)]p'(t) = iB(t)v'(t) \tag{2.6}$$

where $B(t)=e^{-t}/(1-e^{-it})$, $v(t) = i\Sigma n e^{int} v_n$ and v_n is some p.m.f.

and max $(k,\ell)=1$. Let N be discrete r.v. independent of M and $Z = MN$. Define N^+ such that $P|N^+=i|=p|N=i|/p|N > 1|$ and N^- such that $p|N^-=i|=p|N=-i|/p|N \geq -1|$, $i \varepsilon I_+$.

(i) If $\min(k,\ell) = 0$, then Z is unimodal iff both N^+ and N^- are unimodal about 1.

(ii) If $\min(k,\ell) = 1$, then Z is unimodal iff N is unimodal about 0 or 1.

Theorem 3.2: Assume M is uniform r.v. on $\{-k,\ldots,\ell\}$ such that k, $\ell \varepsilon I_+$ and $\max(k,\ell) \geq 2$. Let N be a discrete r.v. independent of M. Then $Z=MN$ is unimodal iff $P||N| \leq 1| = 1$.

The negation of the discrete counterpart of Khintchine-Isii result is based on the fact that for $\ell > 2$, large prime numbers receive smaller mass than that non-prime numbers get from the distribution of Z and the unimodality condition forces it to be zero, Dharmadhikari and Jogdeo (1976).

4. CHARACTERIZATION OF STRONG UNIMODALITY

Ibragimov's characterization of strong unimodality, i.e. log-concavity of the density function $f(x)$ of a r.v. X, say, is expressed through the relation

$$\log f(\lambda x_1+(1-\lambda)x_2) \geq \lambda \log f(x_1)+(1-\lambda) \log f(x_2)$$

where $x_1 < x_2$ and $\lambda \varepsilon (0,1)$. This result is proved in the discrete case by Keilson and Gerber (1971) as follows.

Theorem 4.1: A distribution p_n, $n \in I$ is strognly unimodal iff its p.m.f. p_n satisfies other publications.

5. α-UNIMODAL DISTRIBUTIONS

Chung (1953), see also Chung's comments in Gnedenko and Kolmogrov (1954), has pointed out that the convolution of two continuous unimodal distributions may not be unimodal. The convolution of two symmetric unimodal distributions is symmetric unimodal, Wintner (1938). The class of strong unimodal distributions (either discrete or continuous) is closed under convolution. Dharmdhikari and Jogdeo (1974) have shown by example that there exists a unimodal density function such that its convolution with symmetric non-log-concave unimodal density function give a non-unimodal density. This raises a question whether the symmetry assumption can help in preserving unimodality under convolution or not. Olshen and Savage (1970) have observed Khintchine's characterization of continuous unimodality as a consequence of the integral formulation of the Krein-Milman Theorem. Their definition of α-unimodality can be stated as: A r.v. X is α-unimodal about zero iff $t^{\alpha}Ef(tX)$ is non-decreasing in t for $t > 0$ for every bounded, non-negative, Borel measurable f defined on R. It is noted that the parameter has be non-negative. The ordinary unimodality occurs when $\alpha = 1$. In other words relation (2.1) can be written in the form

$$f(t) = \int_0^1 g(tu)\, du \qquad (5.1)$$

Thus, α-unimodality of some density function f can be characterized if its ch.f. f(t) is represented by discrete case. We have tried to impose the formulation of continuous α-unimodal d.f. whose ch.f. is given by (5.4) on expressions of discrete unimodality given by Theorems 2.4 and 2.5 but it leads to a contradiction especially when one is interested to express this kind of unimodality in terms of p.m.fs.

5. AN APPLICATION

We consider the discrete counterparts of the problems of matching in paired comparisons which was discussed by Hodges and Lehman (1954). Consider 2n subjects divided into n pairs within each pair a treatment is assigned at random to one of the subjects while the other is used as a control. It is known that one simple design to test the effect of treatment in this problem is the method of matching pairs. Let H_0 be the null hypothesis that the treatment has no effect and H_1 be the alternative hypothesis that the treatment causes a specified positive effect. Assume the scores of the first and the second subjects in any pair are discrete r. vs. U and V respectively. Thus under H_0, one gets U=a+X, V=b+Y where X and Y are i.i.d. r.rs. according to a distribution $P_n, n\varepsilon I$. But under H_1 additional quantity t, say, is added to the score of the treated subject. Denote by D_n,n I to a discrete distribution of N=Y - X. Assume that the treatment is applied to either first or second subject of the pair with probability ½ . Let the superscript * denote the treated subject score. Thus under H_0

$$P(U^* > V \text{ or } V^* > U) = \tfrac{1}{2} \tag{5.1}$$

which means that the power test increases by mismatching. This leads to a choice of p_n to be non-unimodal. It has been proved by Hodges and Lehman (1955) and Dharmadhikari and Jogdeo (1983) that a symmetrized continuous d.f of a unimodal d.f. is again unimodal, i.e. if F is continuous d.f. and a d.f. F is defined by $\tilde{F}(-x)=1-F(x)$, then $F^S = F*\tilde{F}$ is called the symmetrized d.f. of F. For brevity, we would conjucture that such result is true for the discrete case.

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