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MULTISTATE COHERENT SYSTEMS OF ORDER k

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Abstract—Recently the theory of multistate reliability models has been developed to cope with many real-life situations. The present paper introduces multistate coherent systems of order k, where k refers to the number of levels for which the underlying components of the system are relevant. It is shown that many multistate systems introduced earlier by different authors are included in the class of multistate coherent systems of order k. The dual class, k-parallel and k-series systems are introduced and studied. The main structural properties of this class, which are analogous to the well-known results for coherent systems, are established.

1. INTRODUCTION

In reliability theory the main problem is to determine the relationship between the reliability of a complex system and the reliabilities of its components. Most of the literature in reliability deals with the binary theory where a system and its components are in either of two states, functioning or failed, see Barlow and Proschan [1] and references therein.

Recently attention has been given to generalizing the binary theory to the theory of multistate where the system and its components assume a whole range of levels of performance ranging from perfect functioning to complete failure. The main contributions to the theory of multistate are due to Barlow and Wu [2], El-Neweihi *et al.* [3], Ross [4] and Natvig [5]. Ebrahimi [6] and El-Neweihl and Proschan [7] have stressed the relevancy condition of multistate systems. Block *et al.* [8] and Abouammoh *et al.* [9] have introduced structure functions of multistate coherent systems based on L-superadditive (or subadditive) and Schur-concave (or convex), respectively.

The basic notion of multistate theory is the relevancy of the components which relates their performance to the performance of their system. In this paper a new class of multistate coherent systems, namely the class of multistate coherent systems of order k, is introduced, where k refers to the minimum number of levels of performance for which the components are relevant.

In Section 2, we present the existing definition of multistate systems, notation and terminology. In Section 3 the new generalized class of multistate coherent systems of order k is introduced and its relations to other multistate systems and its dual class are studied. Section 4 contains basic structural properties of the class of multistate coherent systems of order k. The relations of some of its subclasses to the recent class of multistate systems introduced by Block *et al.* [8] are investigated.

2. NOTATION, DEFINITIONS AND PRELIMINARIES

The vector $\mathbf{x} = (x_1, \ldots, x_n)$ denotes the vector of states of components $1, \ldots, n$.

$$(j_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_n), \quad \text{where } j = 0, 1, \ldots, M.$$

$$(\cdot_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n).$$

$$\mathbf{j} \equiv (j, \ldots, j), \quad \text{where } j = 0, 1, \ldots, M.$$

$$x \vee y \equiv \max(x, y)$$

$$\mathbf{x} \vee \mathbf{y} \equiv (x_1 \vee y_1, \ldots, x_n \vee y_n)$$

$$x \wedge y \equiv \min(x, y)$$

$$\mathbf{x} \wedge \mathbf{y} \equiv (x_1 \wedge y_1, \ldots, x_n \wedge y_n).$$

$\mathbf{x} \geqslant \mathbf{y}$ means $x_i \geqslant y_i$ for $i = 1, \ldots, n$.

$\mathbf{x} > \mathbf{y}$ means $x_i \geqslant y_i$ for $i = 1, \ldots, n$ and $x_i > y_i$ for some i.

Increasing and decreasing are used in place of nondecreasing and nonincreasing, respectively.

When we say $\varnothing(x_1, \ldots, x_n)$ is increasing (decreasing), we mean $\varnothing$ is increasing (decreasing) in each argument.

A basic notion in the theory of binary coherent systems is the structure function $\varnothing: \{0, 1\}^n \to \{0, 1\}$ that determines the state of the system in terms of its components. The binary system is said to have coherent structure if, see Barlow and Proschan [1, p. 6];

(1) the function $\varnothing$ is increasing,
(2) for each i there exists a vector $(\cdot_i, \mathbf{x})$ such that

$$\varnothing(0_i, \mathbf{x}) < \varnothing(1_i, \mathbf{x}).$$

Condition (1) states that improving the component performance must not degrade the system performance. Condition (2) is the relevancy condition that eliminates the effect of irrelevant components on system performance.

Most researchers in multistate theory have considered the structure function $\varnothing : S^n \to S$, where $S = \{0, 1, \ldots, M\}$ is the set representing levels of performance ranging from perfect functioning M to complete failure 0. Here M is a finite integer. Condition (1) is extended simply by requiring $\varnothing(\mathbf{x})$ to be increasing in $\mathbf{x}$. However, condition (2) has been extended by researchers in many different ways each leading to a distinct class of multistate coherent systems.

In the remaining of this section we review briefly the main existing classes of multistate coherent systems, which are related to the multistate coherent systems of order k, for easy reference.

The multistate system is determined by the multistate structure function

$$\varnothing : S^n \to S, \qquad (1)$$

which is assumed to satisfy some reasonable conditions. These conditions are usually encountered in practice. We begin these conditions by the following.

Definition 2.1

A system of n components with structure $\varnothing$ is called a monotone multistate system (MMS) if

(1) $\varnothing$ is increasing
(2) $\varnothing(\mathbf{j}) = j,\ j = 0, \ldots, M$.

Condition (1) states that improving the performance of any component should not harm the system performance. Condition (2) is a generalization of $\varnothing(\mathbf{1}) = 1$ and $\varnothing(\mathbf{0}) = 0$, that is if all components are at level j of performance than so is the system, where $j = 0, 1, \ldots, M$. In the following definition several multistate systems are presented.

Definition 2.2

An MMS with structure function $\varnothing : S^n \to S$ is called:

(1) EPS system if $\forall i, \forall j \geq 1, \exists(\cdot_i, \mathbf{x}) \in S^n$ such that

$$\varnothing(j_i, \mathbf{x}) = j \quad \text{and} \quad \varnothing(l_i, \mathbf{x}) \neq j, \quad \forall l \neq j. \qquad (2)$$

This system was introduced by El-Neweihi *et al.* [3].

(2) NA system if $\forall i, \forall j \geq 1, \exists(\cdot_i, \mathbf{x}) \in S^n$ such that

$$\varnothing(j_i, \mathbf{x}) \geq j \quad \text{and} \quad \varnothing((j-1)_i, \mathbf{x}) < j. \qquad (3)$$

This system is due to Natvig [5].

(3) GR1 system if $\forall i, \forall j \geq 1, \exists(\cdot_i, \mathbf{x}) \in S^n$ such that

$$\varnothing(j_i, \mathbf{x}) > \varnothing((j-1)_i, \mathbf{x}). \qquad (4)$$

This system is due to Griffith [10].

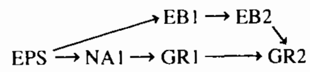

Fig. 1.

(4) EB1 system if $\forall i, \exists j \geq 1, \exists(\cdot_i, \mathbf{x}) \in S^n$ such that

$$\varnothing(j_i, \mathbf{x}) = j \quad \text{and} \quad \varnothing(l_i, \mathbf{x}) \neq j, \quad \forall l \neq j.$$

This system is due to Ebrahimi [6].

(5) EB2 system if $\forall i, \exists j \geq 1, \exists(\cdot_i, \mathbf{x}) \in S^n$ such that

$$\varnothing(j_i, \mathbf{x}) \geq j \quad \text{and} \quad \varnothing((j-1)_i, \mathbf{x}) < j.$$

This system is due to Ebrahimi [6].

(6) GR2 system if $\forall i, \exists j \geq 1, \exists(\cdot_i, \mathbf{x}) \in S^n$ such that

$$\varnothing(j_i, \mathbf{x}) \neq \varnothing(l_i, \mathbf{x}) \quad \text{for some } l \neq j. \qquad (5)$$

This system is due to Griffith [10],

where $i = 1, \ldots, n, \quad j = 0, \quad 1, \ldots, M$.

It can be easily verified that when $M = 1$ all multistate systems given by definition 2.2 are reduced to the binary coherent system.

The implications between the classes of these multistate systems can be summarized in Fig. 1. One can establish counterexamples to show that relationships in Fig. 1 are proper implications.

3. THE CLASS OF MULTISTATE COHERENT SYSTEMS OF ORDER k

In this section we introduce new classes of multistate coherent systems of order k. The basic feature of this class is that its order k depends on the number of levels to which the underlying components are relevant. This class includes the classes of coherent systems introduced in Section 2.

Next we express the structure function $\varnothing$ in terms of binary functions.

Definition 3.1

The binary function $\varnothing_j : S^n \to \{0, 1\}$ of an MMS structure function $\varnothing : S^n \to S$ is given by

$$\varnothing_j(\mathbf{x}) = \begin{cases} 1 & \text{if } \varnothing(\mathbf{x}) \geq j \\ 0 & \text{O.W.,} \end{cases} \qquad (6)$$

for all $j \geq 1$ and $\mathbf{x} \in S^n$.

Note that $\varnothing_j$ is a decreasing function in j, that is

$$\varnothing_j(\mathbf{x}) \geq \varnothing_{j+1}(\mathbf{x}), \quad \forall \mathbf{x} \in S^n.$$

Also $\varnothing(\mathbf{x})$ can be represented in terms of $\varnothing_j$, $j \geq 1$ as follows:

$$\varnothing(\mathbf{x}) = \sum_{j=1}^{M} \varnothing_j(\mathbf{x}), \quad \forall \mathbf{x} \in S^n.$$

A general relevancy condition of a component to an MMS with respect to level j, $j \geq 1$, is now introduced.

Definition 3.2

Given an MMS with structure function $\varnothing : S^n \to S$, let A_j and B_j be two sets of levels of performance associated with levels j, $j \in S$. Then component i, $i = 1, \ldots, n$, is said to be relevant to the system, whose structure function is $\varnothing$, with respect to level $j \geqslant 1$ if there exists a vector $(\cdot_i, \mathbf{x}) \in S^n$ such that for some $k \in A_j$ and for some $l \in B_j$

$$|\varnothing_l(j_i, \mathbf{x}) - \varnothing_l(k_i, \mathbf{x})| = 1. \tag{7}$$

It is shown in the following proposition that the relevancy condition (7) is equivalent to the weak coherency condition given by Griffith. However, the formulation of condition (7) is more powerful. This property comes from the fact that the relevancy conditions of El-Neweihi *et al.* [3], Natvig [5], and El-Neweihi and Proschan [7] can be viewed as the special cases of condition (7).

Remark 3.3

(1) Condition (7) is equivalent to the weak coherency condition (5), i.e.

$$\varnothing(j_i, \mathbf{x}) \neq \varnothing(l_i, \mathbf{x}) \quad \text{for some } l \neq j.$$

(2) Setting $A_j = \{0\}$ and $B_j = \{1, \ldots, M\}$ in eqn (7), we get

$$\varnothing_l(j_i, \mathbf{x}) - \varnothing_l(0_i, \mathbf{x}) = 1, \quad \text{for some } l \geqslant 1,$$

which is equivalent to the relevancy condition given by El-Neweihi and Proschan [7].

(3) Setting $A_j = \{j - 1\}$ and $B_j = \{1, \ldots, M\}$ in eqn (7), we get

$$\varnothing_l(j_i, \mathbf{x}) - \varnothing((j-1)_i, \mathbf{x}) = 1, \quad \text{for some } l \geqslant 1, \tag{8}$$

which is equivalent to condition (4) due to Griffith [10].

(4) Setting $A_j = \{j - 1\}$ and $B_j = \{j\}$ in eqn (7), we get

$$\varnothing_j(j_i, \mathbf{x}) - \varnothing_j((j-1)_i, \mathbf{x}) = 1.$$

This condition is equivalent to condition (3) due to Natvig [6].

(5) Let $B_j = \{j, j+1\}$, by letting eqn (7) hold in the two special cases:

(a) $A_j = \{j - 1\}$ and
(b) $A_j = \{j + 1\}$.

Simultaneously we get

$$\varnothing_j(j_i, \mathbf{x}) - \varnothing_j((j-1)_i, \mathbf{x}) = \varnothing_{j+1}((j+1)_i, \mathbf{x}) - \varnothing_{j+1}(j_i, \mathbf{x}) = 1.$$

This is equivalent to condition (1) due to El-Neweihi *et al.* [3].

Next, we give a definition for a general class of multistate coherent systems. This definition relates the degree of coherency of the system to the number of levels of the system to which each component is relevant.

Definition 3.4

An MMS with structure function $\varnothing : S^n \to S$ is said to be a multistate coherent system of order k (MCS-k), $k = 1, \ldots, M$, if every component i, $i = 1, \ldots, n$, is relevant to at least k levels of performance of the system $\varnothing$ according to the relevancy condition (8).

Notice that an MCS-k is also an MCS-l where $1 \leqslant l \leqslant k$. The class of multistate coherent systems of order k includes most of the classes of multistate coherent systems given in definition 2.2 as subclasses.

Theorem 3.5

Let $\varnothing : S^n \to S$ be the structure function of an MCS-k, then

(1) an MCS-1 is equivalent to a GR2, i.e. an MCS-1 is a multistate weakly coherent system,

(2) by letting $l = j$ in condition (8), an MCS-1 and an MCS-M are an EB2 and an NA, respectively,

(3) an MCS-M is equivalent to a GR1.

Proof. By comparing definitions (2.2) and (3.4) and using remark (3.3), we get the desired result. □

El-Neweihi *et al.* [3] have defined the dual structure function $\varnothing^D$ of the structure $\varnothing$ by

$$\varnothing^D(\mathbf{x}) = M - \varnothing(\mathbf{M} - \mathbf{x}), \quad \forall \mathbf{x} \in S^n,$$

where $\mathbf{M} - \mathbf{x} = (M - x_1, \ldots, M - x_n)$.

For the binary function $\varnothing_j$, we give its corresponding dual by the following result.

Proposition 3.6

The dual binary structure function $\varnothing_j^D$ of $\varnothing^D$, which is given by

$$\varnothing_j^D(\mathbf{x}) = \begin{cases} 1 & \text{if } \varnothing^D(\mathbf{x}) \geqslant j \\ 0 & \text{O.W.}, \end{cases}$$

where $j = 1, \ldots, M$ and $\mathbf{x} \in S^n$, can be expressed in terms of $\varnothing_j$ as follows

$$\varnothing_j^D(\mathbf{x}) = 1 - \varnothing_{M-j+1}(\mathbf{M} - \mathbf{x}), \quad j = 1, \ldots, M, \quad \mathbf{x} \in S^n.$$

Proof. Suppose that $\varnothing^D(\mathbf{x}) \geqslant j$. This implies $\varnothing(\mathbf{M} - \mathbf{x}) < M - j + 1$ or $\varnothing_{M-j+1}(\mathbf{M} - \mathbf{x}) = 0$ and hence $\varnothing_j^D(\mathbf{x}) = 1 - \varnothing_{M-j+1}(\mathbf{M} - \mathbf{x}) = 1$. Similarly one can show that if $\varnothing^D(\mathbf{x}) < j$ then $\varnothing_j^D(x) = 1 - \varnothing_{M-j+1}(\mathbf{M} - \mathbf{x}) = 0$. □

The following proposition shows that a structure function $\varnothing$ is MCS-k if and only if $\varnothing^D$ is MCS-k.

Proposition 3.7

The structure function $\varnothing$ is an MCS-k iff $\varnothing^D$ is an MCS-k, $k = 1, \ldots, M$.

Proof. Suppose $\varnothing$ is an MCS-k structure function, $k = 1, 2, \ldots, M$. Note that $\varnothing^D$ is increasing and $\varnothing^D(j) = M - \varnothing(\mathbf{M} - \mathbf{j}) = j$, $j = 0, \ldots, M$. Now if component i, $i = 1, \ldots, n$ is irrelevant to $\varnothing$ with respect to level $j \geq 1$, then there exist $(\cdot_i, \mathbf{x}) \in S^n$ such that

$$\varnothing_l(j_i, \mathbf{x}) - \varnothing_l((j-1)_i, \mathbf{x}) = 1, \quad \text{for some } l \geq 1.$$

Thus

$$\begin{aligned}\varnothing^D_{M-l+1}((M-j+1)_i, \mathbf{x}) &- \varnothing^D_{M-l+1}((M-j)_i, \mathbf{x}) \\ &= 1 - \varnothing_l((j-1)_i, \mathbf{x}) - (1 - \varnothing_l(j_i, \mathbf{x})) \\ &= \varnothing_l(j_i, \mathbf{x}) - \varnothing_l((j-1)_i, \mathbf{x}) \\ &= 1.\end{aligned}$$

Therefore component i is relevant to $\varnothing^D$ with respect to level $M - l + 1$. Hence if a component is relevant to k levels of $\varnothing$, it is relevant to k levels of $\varnothing^D$. Hence $\varnothing$ is MCS-k. Conversely, let $\varnothing^D$ be MCS-k. Since $[\varnothing^D(x)]^D = M - \varnothing^D(\mathbf{M} - \mathbf{x}) = \varnothing(\mathbf{x})$ and, since the dual of MCS-k is an MCS-k, the desired result follows. □

If $\varnothing \in$ MCS-k where $k = 1, \ldots, M$ then each component is relevant to at least one level performance of $\varnothing$. One may enquire if it is possible to have a level of performance of $\varnothing$ such that no component is relevant to that level. The following theorem shows that this is not possible. In fact it shows that any system is an MMS if for each level of performance there exists at least one component which is relevant to that level.

Theorem 3.8

Let $\varnothing$ be a monotone structure function. Then $\varnothing$ is an MMS iff $\forall j \in S$, $\exists i (i = 1, \ldots, n)$ and $(\cdot_i, \mathbf{x}) \in \{j-1, j\}^{n-1}$ such that

$$\varnothing(j_i, \mathbf{x}) > \varnothing((j-1)_i, \mathbf{x}).$$

Proof. Suppose that $\varnothing$ is an MMS, i.e. $\varnothing(\mathbf{j}) = j$. Assume that $\exists j \geq 1$ such that $\forall i$, $\forall (\cdot_i, \mathbf{x}) \in \{j-1, j\}^{n-1}$:

$\varnothing(j_i, \mathbf{x}) = \varnothing((j-1)_i, \mathbf{x})$, then it follows that

$\varnothing(\mathbf{j}) = \varnothing((\mathbf{j}-\mathbf{1}))$, which is a contradiction.

Now, suppose that $\forall j \geq 1, \exists i, (\cdot_i, \mathbf{x}) \in \{j-1, j\}^{n-1}$ such that $\varnothing(j_i, \mathbf{x}) > \varnothing((j-1)_i, \mathbf{x})$, then $\varnothing(\mathbf{j}) > \varnothing((\mathbf{j}-\mathbf{1}))$. Hence $0 \leq \varnothing(\mathbf{0}) < \varnothing(\mathbf{1}) < \cdots < \varnothing(\mathbf{M}) \leq M$, thus $\varnothing(\mathbf{j}) = j$, $j = 0, 1, \ldots, M$, i.e. $\varnothing$ is an MMS. □

4. STRUCTURAL PROPERTIES

In this section the main structural properties of the MCS-k are investigated. In particular the well-known principle, used by design engineers, namely redundancy at the component level, is preferable to redundancy at the system level and is explored under MCS-k structures. This leads to two classes of multistate structures, the k-parallel and k-series. Some properties of these two classes are studied and their relations to other recent classes of multistate systems are determined.

The principle of redundancy is presented in mathematical form as follows.

Theorem 4.1

Let $\varnothing$ be a structure function of an MCS-k. Then

(1) $\varnothing(\mathbf{x} \amalg \mathbf{y}) \geq \varnothing(\mathbf{x}) \amalg \varnothing(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in S^n$,
(2) $\varnothing(\mathbf{x} \sqcap \mathbf{y}) \leq \varnothing(\mathbf{x}) \sqcap \varnothing(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in S^n$.

Proof. (1) Since $\varnothing$ is increasing, $\mathbf{x} \amalg \mathbf{y} \geq \mathbf{x}$ and $\mathbf{x} \amalg \mathbf{y} \geq \mathbf{y}$, then $\varnothing(\mathbf{x} \amalg \mathbf{y}) \geq \varnothing(\mathbf{x})$ and $\varnothing(\mathbf{x} \amalg \mathbf{y}) \geq \varnothing(\mathbf{y})$. This implies that $\varnothing(\mathbf{x} \amalg \mathbf{y}) \geq \varnothing(\mathbf{x}) \amalg \varnothing(\mathbf{y})$.

(2) can be similarly proven. □

It was shown in the binary model that the equality in parts (1) and (2) of theorem 4.1 holds, if $\varnothing$ is parallel and series, i.e.

$$\varnothing(\mathbf{x}) = \max_{1 \leq i \leq n} x_i$$

and

$$\varnothing(\mathbf{x}) = \min_{1 \leq i \leq n} x_i,$$

respectively. For the MCS-k class a different approach for defining more general parallel and series systems is adopted.

Now we introduce the following concept of parallel (series) systems.

Definition 4.2

Let $\varnothing$ be a structure function of an MCS-k. The MCS-k is said to be a k-parallel (k-series) structure, $k = 1, \ldots, M$, if

$$\begin{aligned}\varnothing(\mathbf{x} \amalg \mathbf{y}) &= \varnothing(\mathbf{x}) \amalg \varnothing(\mathbf{y}) (\varnothing(\mathbf{x} \sqcap \mathbf{y}) \\ &= \varnothing(\mathbf{x}) \sqcap \varnothing(\mathbf{y})),\end{aligned} \tag{9}$$

for every $\mathbf{x}, \mathbf{y} \in S^n$.

Next we prove the following result, which is in line with the traditional parallel and series property.

Theorem 4.3

Let $\varnothing$ be a structure function of an MCS-k. Then $\varnothing$ is k-parallel (k-series) if $\varnothing^D$ is k-series (k-parallel).

Proof. Suppose $\varnothing$ is k parallel. Then

$$\begin{aligned}\varnothing^D(\mathbf{x} \sqcap \mathbf{y}) &= M - \varnothing[\mathbf{M} - (\mathbf{x} \sqcap \mathbf{y})] \\ &= M - \varnothing[(\mathbf{M} - \mathbf{x}) \sqcap (\mathbf{M} - \mathbf{y})] \\ &= M - [\varnothing(\mathbf{M} - \mathbf{x}) \sqcap \varnothing(\mathbf{M} - \mathbf{y})] \\ &= [M - \varnothing(\mathbf{M} - \mathbf{x})] \sqcap [M - \varnothing(\mathbf{M} - \mathbf{y})] \\ &= \varnothing^D(\mathbf{x}) \sqcap \varnothing^D(\mathbf{y}).\end{aligned}$$

Thus $\varnothing^D$ is k-series. The reverse direction is straight forward and the other part of the theorem can be shown by using a similar argument. □

In the following result the traditional parallel and series structures are characterized within the k-parallel and k-series structures, respectively.

Theorem 4.4

Let $\emptyset$ be a multistate structures of MCS-k. Then

(1) $\emptyset$ is M-parallel if $\emptyset(\mathbf{x}) = \max_{1 \leqslant i \leqslant n} x_i$, $\forall \mathbf{x} \in S^n$,

(2) $\emptyset$ is M-series if $\emptyset(\mathbf{x}) = \min_{1 \leqslant i \leqslant n} x_i$, $\forall \mathbf{x} \in S^n$.

Proof. Note that an M-parallel (M-series) structure is just GR1 system which satisfies condition (9). Now, the desired results can be proved by using similar argument of Griffith. □

Note that for $k = 1, \dots, M-1$ theorem 4.4 does not hold, as we see in the following example.

Example 4.5

Consider the MCS-2 structure given $\emptyset$ in the following table:

x_1 \ x_2	0	1	2	3
0	0	1	1	2
1	0	1	1	2
2	2	2	2	2
3	3	3	3	3

Since

$$\emptyset(1,0) = 0, \quad \emptyset(\mathbf{X}) \neq \max_{1 \leqslant i \leqslant 2} X_i.$$

But $\emptyset(\mathbf{X} \pi \mathbf{Y}) = \emptyset(\mathbf{X}) \pi \emptyset(\mathbf{Y})$, $\forall \mathbf{X}, \mathbf{Y} \in S^2$, i.e. $\emptyset$ is 2-parallel.

We call the function $\emptyset$ permutation invariant if $\emptyset(\mathbf{x}) = \emptyset(\Pi(\mathbf{x}))$, where Π is any permutation of the components of the state vector $\mathbf{x}$. Now, we prove the following.

Theorem 4.6

Let $\emptyset$ be an MCS-K, $1 \leqslant k \leqslant M$. Then:

(1) $\emptyset(\mathbf{x}) = \max_{1 \leqslant i \leqslant n} x_i$, $\forall \mathbf{x} \in S^n$ if $\emptyset$ is k-parallel and permutation invariant,

(2) $\emptyset(\mathbf{x}) = \min_{1 \leqslant i \leqslant n} x_i$, $\forall \mathbf{x} \in S^n$ if $\emptyset$ is k-series and permutation invariant.

Proof. (1) Suppose that $\emptyset(\mathbf{x}) = \max_{1 \leqslant i \leqslant n} x_i$, $\forall \mathbf{x} \in S^n$ then it is clear that $\emptyset$ is k-parallel and permutation invariant. Conversely, if $\emptyset$ is k-parallel and permutation invariant, then

$$\begin{aligned} j = \emptyset(\mathbf{j}) &= \emptyset[(j_1, \mathbf{0}) \vee (j_2, \mathbf{0}) \vee \cdots \vee (\mathbf{0}, j_n)] \\ &= \emptyset(j_1, \mathbf{0}) \vee \cdots \vee \emptyset(\mathbf{0}, j_n) \\ &= \max_{1 \leqslant i \leqslant n} \emptyset(j_i, \mathbf{0}). \end{aligned}$$

But $\emptyset(j_i, \mathbf{0})$ is the same for all i (since $\emptyset$ is permutation invariant) so,

$$\emptyset(j_i, \mathbf{0}) = j, \quad \forall i, \quad \forall j,$$

so

$$\begin{aligned} \emptyset(\mathbf{x}) &= \max_{1 \leqslant i \leqslant n} \emptyset(x_i, \mathbf{0}) \\ &= \max_{1 \leqslant i \leqslant n} x_i, \quad \forall \mathbf{x} \in S^n. \end{aligned}$$

(2) The proof is similar. □

The K-parallel (k-series) structures are contained properly in the L-superadditive (L-subadditive) classes of multistate structures due to Block *et al.* [8]. To show that we give the following.

Definition 4.7

An increasing structure function is called L-superadditive (LSP) if

$$\emptyset(\mathbf{x} \amalg \mathbf{y}) + \emptyset(\mathbf{x} \Pi \mathbf{y}) \geqslant \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in S^n.$$

If the reverse inequality holds, $\emptyset$ is called L-subadditive (LSB).

Note that $\emptyset$ is LSP iff $\emptyset^D$ is LSB.

Now we prove the following.

Theorem 4.8

If the structure function $\emptyset$ is k-series (k-parallel), then $\emptyset$ is LSP (LSB).

Proof. Let $\emptyset$ be k-series, then

$$\emptyset(\mathbf{x} \Pi \mathbf{y}) = \emptyset(x) \Pi \emptyset(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in S^n.$$

Without loss of generality one may assume $\emptyset(\mathbf{x}) \leqslant \emptyset(\mathbf{y})$, then

$$\begin{aligned} \emptyset(\mathbf{x} \Pi \mathbf{y}) &= \emptyset(\mathbf{x}) \\ &= \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}) - \emptyset(\mathbf{y}) \\ &\geqslant \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}) - \emptyset(\mathbf{x} \Pi \mathbf{y}), \end{aligned}$$

i.e.

$$\emptyset(\mathbf{x} \Pi \mathbf{y}) + \emptyset(\mathbf{x} \Pi \mathbf{y}) \geqslant \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}), \text{ or } \emptyset \text{ is LSP}.$$

Using theorem 4.3 and the fact that $\emptyset$ is LSP if $\emptyset^D$ is LSB the remaining proof of the theorem can be established. □

The following example shows that an LSP (LSB) structure function may not be a k-series (k-parallel) structure.

Example 4.9

Consider the structure function $\varnothing$ given by the following table:

x_2 \ x_1	0	1	2
0	0	1	1
1	0	1	1
2	1	2	2

Note that $\varnothing$ is an MCS-1. Let $\mathbf{X} = (1, 0)$ and $\mathbf{Y} = (0, 2)$. Since

$$\varnothing(\mathbf{x} \amalg \mathbf{y}) > \varnothing(\mathbf{x}) \amalg \varnothing(\mathbf{y})$$

and

$$\varnothing(\mathbf{x} \Pi \mathbf{x}) < \varnothing(\mathbf{x}) \Pi \varnothing(\mathbf{y}),$$

$\varnothing$ is neither k-parallel nor k-series. Furthermore note that

$$\varnothing(\mathbf{x} \amalg x) + \varnothing(\mathbf{x} \amalg \mathbf{y}) = \varnothing(\mathbf{x}) + \varnothing(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in S^n,$$

thus $\varnothing$ is an LSP and an LSB, simultaneously.

Fortunately, when the order of the system is M, the M-series (M-parallel) and LSP (LSB) structures are equivalent. This result was proved by El-Neweihi and Meng [11].

Theorem 4.10

Let $\varnothing: S^n \to S$ be an MCS-M. Then M-series (M-parallel) and LSP (LSB) structures are equivalent.

To find the number of min path (max cut) vectors to a particular level in the k-series (k-parallel) systems we need to present the following definition.

Definition 4.11

A state vector $\mathbf{x}$ is called a min path vector to level $j, j = 1, \ldots, M$ if $\varnothing_j(\mathbf{x}) = 1$ and $\varnothing_j(\mathbf{y}) = 0$, for every $\mathbf{y} < \mathbf{x}$. Similarly a state vector $\mathbf{x}$ is said to be max cut to level j if $\varnothing_{j+1}(\mathbf{x}) = 0$ and $\varnothing_{j+1}(\mathbf{y}) = 1$ for every $\mathbf{x} < \mathbf{y}$, where $\varnothing_j(\cdot)$ is given by definition 3.1.

Using the definitions of min path and max cut vectors one can establish the following result for the class of MMS.

Theorem 4.12

Let $\varnothing$ be a multistate structure function. $\mathbf{x}$ is a min path (max cut) vector to level j of $\varnothing$ if $\mathbf{M} - \mathbf{x}$ is a max cut (min path) vector to level $M - j$ of $\varnothing^D$.

Proof. Suppose $\mathbf{x}$ is a min path vector to level j of $\varnothing$. Then $\varnothing_j(\mathbf{x}) = 1$ and $\varnothing_j(\mathbf{y}) = 0$, $\forall \mathbf{y} < \mathbf{x}$. This implies

$$\varnothing^D_{M-j+1}(\mathbf{M} - \mathbf{x}) = 1 - \varnothing_j(\mathbf{x}) = 0.$$

Since $\mathbf{M} - \mathbf{y} > \mathbf{M} - \mathbf{x}$, thus

$$\varnothing^D_{M-j+1}(\mathbf{M} - \mathbf{y}) = 1 - \varnothing_j(\mathbf{y}) = 1.$$

Therefore $\mathbf{M} - \mathbf{x}$ is a max cut to level $M - j$ of $\varnothing^D$. The converse and the other part can be carried out by similar arguments. □

The k-series (k-parallel) structure is shown in the following to have only one min path (max cut) vector for every level of performance.

Theorem 4.13

Let $\varnothing$ be a k-series (k-parallel) structure. Then $\varnothing$ has only one min path (max cut) vector to every level $j, j = 1, \ldots, M$ ($j = 0, \ldots, M - 1$).

Proof. Let $\varnothing$ be a k-series structure. The existence of at least one min path (max cut) vectors is guaranteed by the fact that $\varnothing(\mathbf{j}) = j$, $\forall j = 0, \ldots, M$. Now assume there are two distinct min path vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ to level $j \geqslant 1$, i.e.

$$\varnothing(\mathbf{y}_1) = 1 \quad \text{and} \quad \varnothing(\mathbf{y}'_2) = 0, \quad \forall \mathbf{y}'_1 < \mathbf{y}_1,$$

and

$$\varnothing(\mathbf{y}_2) = 1 \quad \text{and} \quad \varnothing(\mathbf{y}'_2) = 0, \quad \forall \mathbf{y}'_2 < \mathbf{y}_2.$$

Since $\varnothing$ is k-series we have

$$\varnothing(\mathbf{y}_1 \Pi \mathbf{y}_2) = \varnothing(\mathbf{y}_1) \Pi \varnothing(\mathbf{y}_2) \geqslant j,$$

but $\mathbf{y}_1 \Pi \mathbf{y}_2 < \mathbf{y}_1$, which leads to a contradiction.

Similarly an argument can be used to show that the k-parallel structure has only one max cut vector. □

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