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ON THE STRUCTURE AND APPLICATIONS OF INFINITE DIVISIBILITY, STABILITY AND SYMMETRY IN STOCHASTIC INFERENCE

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1. INTRODUCTION

The concepts of infinite divisibility, stability, symmetry and unimodality besides being useful and well-known in probability and statistics, are basic to many inference problems. Some of these concepts are also related to stationary independent stochastic processes, infinitesimal random systems, exchangeable and spherical exchangeable processes, the random sample idea, and nonparametric methods, which are fundamental in theory and applications and occur frequently in many real life situations. Among a wide range of applications, one finds these ideas arising in economic time series, industry, meterology, quantum physics and atomic nuclei, fluid and liquid analyses, signal detection, biology, biomedical and other scientific fields.

It is most common in many spheres that random phenomena take place due to the effect of a large number of causes, where each single cause individually has only an insignificantly small effect on the phenomena, but the total aggregate effect is worth probabilistic-statistical study. Here, one can recall the basic derivations of the Gaussian processes or normality. This brings to mind the concepts of infinitesimal random systems, infinite divisibility, and sums of independent or dependent random variables. There is a large amount of work in this area, for example Lévy (1937), Gnedenko-Kolmogorov (1954), Loève (1963) Lukacs (1970), Feller (1971), Petrov (1975), and many other references contained therein. In a recent paper Steutel (1973) has surveyed some of the work on infinite divisibility during the last fifteen years.

In the problems of filtering, forecasting, prediction, and control, the underlying processes are assumed to be stable in some sense. Also, the concept of symmetry, besides being useful in the above situations, is basic to some nonparametric problems. For estimating unknown probability density functions, one often assumes the underlying family to be 'smooth' in some sense, that is unimodal, symmetric or stable. Finally, in many 'smoothing' operations in mathematical and sometimes even 'nonmathematical' sciences, one employs the above concepts via convolutions, mixing, averaging, or some other suitable transformations. Similarly, these concepts and the ideas of exchangeability and spherical exchangeability, occur in psychological and learning experiments, quantum physics, solar wind and nozal gas flow etc., to the spread of many bacteriological colonies, biological populations, some other growth phenomena, polutants, ambisonic music system and Bayesian statistics, see Ahmad (1974, 1975a). Further, see the role played by unimodality in Barndorff-Nielsen (1976) as applied to the plausibility inference which is in a sense complimentary to likelihood inference and its many variations. Finally, note that the largest family of infinite divisible distributions which includes symmetric stable, stable, some unimodal (strongly unimodal, universal), and many variants of the so called $\mathcal{L}$-class distributions, is itself contained in the totality of limit laws of the Central Limit Problems, see Loève (1963, page 297). Thus, one is in the realm of asymptotic statistics which is playing an ever greater role in the theory and applications of probability and statistics.

The object of this paper is to unify, extend wherever possible, and give interrelationships of the above concepts with their many and varied applications. The effort in this direction is not claimed to be exhaustive and definitive, rather we have tried to deal with those aspects of the above ideas which will be of some interest to applied mathematical statisticians in a broad sense.

2. THE FUNDAMENTAL MEASURE-THEORETIC AND FUNCTIONAL-ANALYTICAL RESULTS

In the sequel, we shall briefly discuss some of the fundamental functional-measure-theoretic results which form the basis for many important concepts in probability theory. These results are well known, though scattered throughout the literature. For brevity and completeness, we restrict our choice to the ideas of basic import, which are used in this paper.

Definition 2.1. A function $h(x)$ $(-a < x < a)$ belongs to the class Σ_a, if it is hermitian positive (that is, $\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} h(x_\alpha - x_\beta)\xi_\alpha\bar{\xi}_\beta \geq 0$, x's real and ξ's complex) in $(-a,a)$ and is continuous at the point $x = 0$.

The following result was obtained by Bochner in 1932 for $a = \infty$ and by Krein (1940) for $a < \infty$.

Lemma 2.1. In order that a function $\phi(x)$ $(-a < x < a)$ may have a representation of the form $\phi(x) = \int e^{ixt}\, dF(t)$, where $F(t)$ is a nondecreasing function of bounded variation, it is necessary and sufficient that $\phi(x) \in \Sigma_a$.

Notice the importance of the above result in the form of characteristic functions in the probability theory and also in Laplace-Fourier-Stieltjes transforms. In fact, we have the following corollary of the above lemma.

Corollary 2.1. The family $\Sigma_\infty = \{\phi\}$, with the normalization $\phi(0) = 1$, coincides with the aggregate of all characteristic functions for all probability distributions.

Definition 2.2. A function $h^*(x)$ $(-a < x < a)$ belongs to the class Σ_a^* $(a \leq \infty)$, if for every positive integer it is the nth root of some function $h(x)$ of the class Σ_a.

Clearly, $\Sigma_a^* \subset \Sigma_a$, and as we shall see later the family Σ_∞^* with the normalization $h^*(0) = 1$ coincides with the class of all infinitely divisible characteristic functions.

Now, let a function $h(\underset{\sim}{x}) = h(x_1, \ldots, x_m) = h(x)$, $x \in E^m(-\infty < x_1, \ldots, x_m < \infty)$ say belongs to the class $\Sigma(E^m)$, if it is continuous at 0 and is hermitian positive, that is, for any positive integer n and any real vectors $x_k (k = 1, \ldots, n)$ $x_k \in E^m$, $\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} h(x_\alpha - x_\beta)\xi_\alpha\bar{\xi}_\beta \geq 0$ for complex ξ's. The following result was originally proved by Bochner in 1933 but a proof can be found in Bochner (1955).

Lemma 2.2. Any function $\phi(x) \in \Sigma(E^m)$ permits the representation $\phi(x) = \int_{E^m} e^{ixt'}\, dF_\mu(t)$, where $F(t)$ is a monotonic point function generated by a certain measure $\mu(A)$, $A \subset E^m$. The measure is uniquely determined by the function $\phi(x)$. Conversely, any function $\phi(x)$ which has the above representation belongs to the class $\Sigma(E^m)$.

Let $G_a = \{g(x)\colon -a < x < a,\ a \leq \infty\}$ be a class of functions such that they are continuous at zero, $g(0) = 0$, $g(-x) = \overline{g(x)}$, and generate a hermitian positive kernel $K(x,y)$ $(0 \leq x, y < a)$ by the formula $K(x,y) = g(x) + g(-y) - g(x-y)$.

Theorem 2.1. (i) If $g(x) \in G_a$, then $\exp(-g(x)) \in \Sigma_a$. More precisely $g(x) \in G_a$ implies $\exp(-g(x)) \in \Sigma_\infty^*$. Note $\Sigma_a^* \subset \Sigma_a$ $(a \leq \infty)$. (ii) For any $h^*(x) \in \Sigma_a^*$, there exists $g(x) \in G_\infty$ such that $h^*(x) = h^*(0)\exp(-g(x))$ $(-a < x < a)$. (iii) Any function $h^*(x) \in \Sigma_a^*$ $(a < \infty)$ can be continued into a function of the class Σ_∞^*. Similarly, any function $g(x) \in G_a$ $(a < \infty)$ can be continued into a function of the class G_∞. (iv) Any function $g(x) \in G_a$ $(a \leq \infty)$ permits the representation

$$g(x) = i\gamma x + \int_{-\infty}^{\infty} \left[1 + \frac{iux}{1+u^2} - e^{iux}\right] u^{-2}\, dF(u), \tag{2.1}$$

$(-a < x < a)$ where γ is real and $F(u)$ is a nondecreasing point function such that $\int_{-\infty}^{\infty}(1+u^2)^{-1}\, dF(u) < \infty$. Here $g(x)$ determines γ and for $a = \infty$, $F(u)$ is determined essentially uniquely. Conversely, any function $g(x)$ which permits the form (2.1) belongs to G_a.

Proof. In cases (i), (iii) and (iv) the proofs can be found in Krein (1944) and Akhiezer-Glazman (1957) with slight modifications. In particular, the first part of (iv) is based on the well known Lévy-Khintchine canonical representation, which we shall discuss in the next section. Since part (ii) of the theorem plays an important role in the later sections, we shall prove it for completeness.

Without loss of generality assume $h^*(0) = 1$. By definition one can find a function $h_n(x) \in \Sigma_a$ for any positive integer n such that $h^*(x) = [h_n(x)]^n$ $(-a < x < a)$. By (iii) the function $h_n(x)$ can be extended into a function $\tilde{h}_n(x) \in \Sigma_\infty$, so we can construct the function $g_n(x) = n[1-\tilde{h}_n(x)]$ which belongs to family G_∞.

Choose $\varepsilon < a/2$ so that for $-\varepsilon \leq x \leq \varepsilon$ the value of $h^*(x)$ is sufficiently close to unity. Let $q(x) = \operatorname{Im} \ln h^*(x)$, then for $-\varepsilon \leq x \leq \varepsilon$, we have:

$$\operatorname{Re}[g_n(x)] = n\operatorname{Re}[1-(h^*(x))^{1/n}] = n[1-|h^*(x)|^{1/n}\cos q(x)n^{-1}],$$

$$\operatorname{Im}[g_n(x)] = -n|h^*(x)|^{1/n}\sin q(x)n^{-1}.$$

Therefore, $|\operatorname{Re}[g_n(x)]| \leq \ln|h^*(x)|^{-1} + (2n)^{-1}q^2(x)$, $|\operatorname{Im}[g_n(x)]| \leq |q(x)|$.

The above inequalities imply that $\max_{0\leq x\leq\varepsilon} |g_n(x)| \leq \delta(\varepsilon)$, and also that for $-\varepsilon^* \leq x^* \leq \varepsilon^*$ $(\varepsilon^* \leq \varepsilon)$ $|\operatorname{Re}[g_n(x^*)]| + |\operatorname{Im}[g_n(x^*)]| \leq \delta(\varepsilon^*)$,

where the quantity, $\delta(\varepsilon^*) = \operatorname{supremum}_{-\varepsilon^* \leq x \leq \varepsilon^*} [\ln|h^*(x)|^{-1} + |q(x)| + q^2(x)/2]$,

is independent of n and goes to zero with ε^*.

Next, one can easily establish the sequence $\{g_n(x)\}$ $(-\infty < x < \infty$, $n = 1, 2, \ldots)$ is equicontinuous and uniformly bounded in every finite interval. Hence, there exists a subsequence $\{g_{n_k}(x)\}_{k=1}^{\infty}$ and such a continuous function $g(x)$ $(-\infty < x < \infty)$ that $\lim_{k\to\infty} g_{n_k}(x) = g(x)$ is uniform in every finite interval. The function $g(x) \in G_\infty$, and also for $-a < x < a$: $g(x) = \lim_{k\to\infty} n_k[1 - (h^*(x))^{1/n_k}] = \ln h^*(x)$, which proves the result.

At this stage, one immediately notices the central role played by the classes of functions Σ, Σ^* and G_a, G_∞ in probability and statistics. Furthermore, these ideas motivate semigroups in probability and functional analysis.

3. THE BASIC CANONICAL REPRESENTATIONS:-

In this section we shall unify the most basic and important canonical representation results. In particular we investigate the role of infinite divisibility including the solution of limit problems for sums of independent random variables.

Definition 3.1. (i) The distribution $F(x)$ is said to be infinitely divisible if for each natural number n there exists $F_n(x)$ such that $F = F_n^{n*}$, the nth fold convolution. (ii) The distribution function F is called stable if, for any a_1, $a_2 > 0$ and any b_1, b_2, there exist $a > 0$ and b such that $F(a_1x+b_1) * F(a_2x+b_2) = F(ax+b)$. (iii) The distribution F is said to be symmetric around c, if for every $d \geqslant 0$, $F(c-d) = 1 - F(c+d)$, that is, $P(X-c\leqslant -d) = P(X-c\geqslant d)$, for every $d\geqslant 0$. (iv) The distribution F is called unimodal if there exists at least one a such that $F(x)$ is convex in $x < a$ and concave in $x > a$.

Let $X_1, \ldots, X_j, \ldots$ be independent identically distributed random variables with distribution F, and set $S_n = \sum_{j=1}^{n} X_j$. In general, it is not possible to find normalizing constants A_n, B_n such that $B_n^{-1}(S_n-A_n)$ converges to any nondegenerate distribution. In fact, there is no subsequence for some distributions such that $B_{n_k}^{-1}(S_{n_k}-A_{n_k})$ converges in distribution. For example, the (infinitely divisible) distribution with characteristic function:

$$\text{(3.1)} \qquad \log \phi(t) = \int_{-\infty}^{-\varepsilon}(\cos t x-1)d(4 \log|x|)^{-1} + \int_{\varepsilon}^{\infty}(\cos tx-1)d(4 \log x)^{-1}$$

see Ibragimov-Linnik (1971, p. 267). It is well known that the general stochastic process X_t in R^m with stationary independent increments has the following canonical representation:

$$\text{(3.2)} \quad \log E \exp[i\langle u, X(t)\rangle] = t[i\langle u, a\rangle - u'Au/2 + \int(\exp(i\langle u, x\rangle)-1-i\langle u, g(x)\rangle)d\mu(x)],$$

where $\langle .,.\rangle$, denotes the inner product, $\int$ denotes integration over the domain $R^m-\{0\}$, $[g(x)]_j$ is 1, −1, x_j according as $x_j \geqslant 1$, $x_j \leqslant -1$, $|x_j| < 1$, x_j is the jth coordinate of x, a is a real vector, A is a symmetric nonnegative definite matrix, and the Lévy measure $\mu(\cdot)$ is a Borel measure on $R^m - \{0\}$ such that $\int|g(x)|^2d\mu(\cdot) < \infty$. Thus, given such a triple (a,A,μ), there exists a corresponding process with stationary independent increments having the representation (3.2). From this representation, if $\int_{|x|>1} |x|d\mu < \infty$, then it is easy to see that $E(X(t)) = t[a+\int(x-g(x))d\mu(x)]$; and also, if $\int|x|^2d\mu(x) < \infty$, then $E|X(t) - EX(t)|^2 = t[\text{trace } (A) + \int|x|^2d\mu(x)]$.

The representation (3.2) has a strong resemblance with the Lévy-Khintchine canonical representation of an infinitely divisible characteristic function. The equivalence of such representations we shall rigorously prove in the next section via the stochastic semigroup theory. But now we give various interrelationships of such representations. The most useful and often quoted is the Lévy-Khintchine representation given below.

Lemma 3.1. A function $\phi(t)$ is an infinitely divisible characteristic function, if and only if, it admits the representation:

$$\text{(3.3)} \qquad \log \phi(t) = iat + \int_{-\infty}^{\infty} (e^{itx} - 1 - \frac{itx}{1+x^2}) \frac{1+x^2}{x^2} dG(x),$$

where a is a real constant, $G(x)$ is nondecreasing bounded left continuous function, and the function under the integral sign is equal to $-t^2/2$ for $x = 0$. This representation is unique if $G(-\infty) = 0$.

Proof. See Gnedenko-Kolmogorov (1954) or Petrov (1975).

Examples 3.1. Clearly given (a,G) one can construct the infinitely divisible distribution via the representation (3.3), and conversely. We now present an expression for the pair (a,G) for some important infinitely divisible distributions. For the normal distribution $N(\mu,\sigma^2)$, $a = \mu$, and $G(x) = 0$, if $x \leq 0$, and $G(x) = \sigma^2$ if $x \geq 0$. For the extended Poisson distribution, $P(\alpha,\beta,\lambda)$ with density function $p(t) = P(X = \alpha + \beta k) = (k!)^{-1} \lambda^k e^{-\lambda}$, $k = 0, 1, \ldots$, one notices that $a = \alpha + (\beta\lambda)(1+\beta^2)^{-1}$, $G(x) = 0$, if $x \leq \beta$, and $G(x) = (\beta^2\lambda)(1+\beta^2)^{-1}$, if $x > \beta$. For the degenerate distribution with characteristic function e^{ict}, $a = c$, the degeneracy point, and $G(x) \equiv 0$. We remark that a distribution F carried by a finite interval is not infinitely divisible except if it is concentrated at one point.

An alternative canonical representation for an infinitely divisible distribution is the so called Lévy representation. This states that a function $\phi(t)$ is an infinitely divisible characteristic function, if and only if, it admits the representation

$$(3.4)\qquad \log \phi(t) = iat - \sigma^2t^2/2 + \int_{-\infty}^{\infty}(e^{itx} - 1 - \frac{itx}{1+x^2})dL(x),$$

where a is a real constant, $\sigma^2 \in R_+^1$, the Lévy function L(x) is nondecreasing in $(-\infty,0)$ and $(0,+\infty)$, and satisfies the condition $L(-\infty) = 0 = L(+\infty)$, also $\int_{-\varepsilon}^{+\varepsilon} x^2dL(x) < \infty$ for every $0 < \varepsilon < \infty$. This representation like (3.3) is unique.

We recall the expressions EX(t) and Var X(t) and the fact that if $E(X^2) < \infty$, $\phi''(0)$ exists. In the later case the converse assertion is also true, that is, a function $\phi(t)$ is an infinitely divisible characteristic function with a finite variance, if and only if, it admits the canonical representation:

$$(3.5)\qquad \log \phi(t) = iat + \int_{-\infty}^{\infty} x^{-2}(e^{itx} - 1 - itx)dK(x),$$

where a is a real constant, the Kolmogorov function K(x) is nondecreasing, and the integrand is defined to be $-t^2/2$ at $x = 0$. In the sequal, the functions $\mu(x)$, G(x), L(x), K(x) appearing in (3.2), (3.3), (3.4), (3.5) are respectively called the Lévy spectral Borel measure, Lévy-khintchine, Lévy, and Kolmogorov functions.

In fact, (3.3) can be identified uniquely by the triple (a,σ^2,L) in (3.4) as follows. Let

$$(3.6)\qquad G(x) = \begin{cases} \int_{-\infty}^{x} y^2(1+y^2)^{-1}dL^-(y), & x < 0 \\ \int_{x}^{\infty} y^2(1+y^2)^{-1}dL^+(y), & x > 0 \end{cases}$$

where L^- and L^+ are increasing continuous functions such that $\int_{-\varepsilon}^{0} y^2dL^-(y) + \int_{0}^{\varepsilon} y^2dL^+(y) < \infty$ and $L^-(-\infty) = L^+(+\infty) = 0$. Then by replacing G(x) in (3.3) by that in (3.6), we get

$$(3.7)\qquad \log \phi(t) = iat - \sigma^2t^2/2 + \int_{-\infty}^{0} (e^{itx}-1 - \frac{itx}{1+x^2})dL^-(x) + \int_{0}^{\infty} (e^{itx} - 1 - \frac{itx}{1+x^2})dL^+(x)$$

which is an alternative Lévy canonical representation equivalent to the form (:

Let the double sequence of r.v.'s $\{X_{jn}, j = 1,2, \ldots\}$ be such that random variables in each row are independent and let F_{n_j}, ϕ_{n_j} be the distribution and characteristic function of X_{n_j}.

<u>Definition 3.2.</u> The double sequence above is said to be <u>uniformly infinites random system</u>, infinitesimal system for short (or holospoudic sequence, see

Chung (1974)), if and only if, $\lim_{n\to\infty} \max_{1\leq j\leq n_j} P(|X_{n_j}| > \varepsilon) = 0$.

A necessary and sufficient condition for this to hold is that for every $t \in R^1$: $\lim_{n\to\infty} \max_{1\leq j\leq n_j} |\phi_{n_j}(t) - 1| = 0$, which is equivalent to the condition that $\lim_{n\to\infty} \max_{1\leq j\leq n_j} \int_{-\infty}^{\infty} x^2(1+x^2)^{-1} dF_{n_j} = 0$.

We denote by the class $\mathcal{L}$ or the $\mathcal{L}$- functions the family of all distribution functions which are limits of distributions of sums $Y_n = B_n^{-1} \sum_{i=1}^{n} X_i - A_n$ where $\{X_n\}$ is a sequence of independent random variables such that $\{X_{n_j} = B_n^{-1} X_j\}$ $(1 \leq j \leq n)$ form an infinitesimal system, and $A_n \in R$, $B_n \in R^+$, the positive real line. The infinite smallness assumption in above is the necessary condition for $F(x)$ defined by above to be infinitely divisible. Now let

Ω_0 = the class of all infinitely divisible distributions;

$\Omega_1 \equiv \mathcal{L}$ = the family of the Lévy-Khintchine and Gnedenko-Kolmogorov $\mathcal{L}$-functions as defined above;

Ω_2 = the family of all stable distributions;

Ω_3 = the class of all symmetric stable distributions.

To give a unified structure and interrelationships of the Ω-families, we need the following result.

Theorem 3.1. Let F be a distribution and ϕ its characteristic function. Then

(a) $F \in \Omega_2$, if and only if, (σ^2, L) in the representations (3.4) or (3.7) satisfy one of the following conditions: (i) $L(x) \equiv 0$, (ii) $\sigma^2 = 0$, $L(x) = c\,|x|^{-\alpha}$ for $x < 0$, $L(x) = -dx^{-\alpha}$ for $x > 0$ $(0 < \alpha < 2$, $c \geq 0$, $d \geq 0$, $c + d > 0)$.

(b) $F \in \Omega_2$, if and only if, ϕ has the representation:

$$\log \phi(t) = iat - c|t|^{\alpha} (1 + i\beta t|t|^{-1} g(\alpha,t)), \tag{3.8}$$

where a is real, $c \geq 0$, $0 < \alpha \leq 2$, $|\beta| \leq 1$, and $g(\alpha,t)$ equals $\tan \pi\alpha/2$ for $\alpha \neq 1$ and $2/\pi \tan |t|$, if $\alpha = 1$.

The proofs of the above results can be found in Gnedenko-Kolmogorov (1954) and Ibragimov-Linnik (1971). On the other hand, an infinitely divisible distribution F belongs to the class $\mathcal{L}$, if and only if, in (3.7) the Lévy spectral function L^+ and L^- (or L in (3.4)) are continuous at every point $x \neq 0$ and have left and right derivatives and $xL^{+'}$, $xL^{-'}$ are nonincreasing, where L' denotes either the left or right derivative. With these considerations, we see that $\Omega_2 \subset \Omega_1 \subset \Omega_0$. Furthermore, if we assume that the X's have the same distribution, then the class $\mathcal{L}^*(\equiv\Omega_1^*)$, the $\mathcal{L}$-class obtained by this extra restriction, is such that $\Omega_1^* \equiv \Omega_2$. In fact, the stable distributions are the simplest version of $\mathcal{L}$-distributions. We remark that the degenerate and normal distributions belong to the class $\mathcal{L}$, but $P(\alpha,\beta,\lambda)$ does not since L is discontinuous at $\beta = 0$. Now, we give a method via the canonical representations by which one can construct an example of a distribution function belonging to Ω_i but not to Ω_{i+1} for $i = 0, 1, 2$, respectively.

Example 3.2. In this example we shall show how one can construct a distribution of class Ω_0 but not an $\mathcal{L}$-function. From (3.7) we choose Lévy spectral functions such that L^+ and L^- are increasing continuous functions and at least one of the $xL^{+'}(x)$, $xL^{-'}(x)$ is increasing. Choose $xL^{-'}(x) = \lambda x$, or zero for $x \leq 1$, or $x > 1$, $L^+(x) = 0$ for $x < 0$ and $a = \lambda \log \sqrt{2}$. Then the corresponding characteristic function can be written as

$$\log \phi(t) = \lambda/it(e^{it}-it-1).$$

This is the simplest version of distribution in Ω_o but not in Ω_1. To obtain an $\mathcal{L}$-distribution function but not stable we choose $xL^{-\prime}(x)$ and $xL^{+\prime}(x)$ as non-increasing, nondecreasing functions of x. For simplicity take $L^-(x) = 0$ for $x < 0$, $L^+(x) = \lambda \log x$ for $x \leqslant 1$ and 0 if $x > 0$ and $a = \lambda \pi/4$. Then

$$\phi(t) = \exp\left[\lambda \int_o^t (e^{ix}-1) \frac{dx}{x}\right]$$

is an $\mathcal{L}$-characteristic function but not stable. To get a characteristic function which belongs to Ω_2 but not to Ω_3, one can proceed similar to the above two examples. It is rather complicated to express stable densities in closed form. However, series expansions are available.

For many other interesting examples of Ω- classes and domains of attraction for the class Ω_2 we refer the reader to Lévy (1937) and Feller (1971). To summarize we have seen that the following is valid.

Theorem 3.2. (i) $\Omega_3 \sqsubset \Omega_2 \sqsubset \Omega_1 \sqsubset \Omega_o$. (ii) If the defining sequence for the class Ω_1 is that of independent identically distributed random variables, then $\Omega_1 \equiv \Omega_2$.

After these preliminaries and some basic canonical representations of the above classes, we shall start on a unified algebraic structure approach, but first a few remarks about the family $\mathcal{L}$ and its subfamilies are in order. Some authors for example Loève (1963, page 322) and Urbanik (1973) have called $\mathcal{L}$-functions as self-decomposable distributions. Here, a distribution F is said to be self-decomposable if for every $\alpha \varepsilon (0,1)$, there exists a characteristic function ϕ_α such that $\phi_F(t) = \phi_F(\alpha t)\ \phi_\alpha(t)$. This definition could be thought of as a necessary and sufficient condition for an $\mathcal{L}$-function. Also, compare this with the definition of a stable distribution of the family $\Omega_2 \sqsubset \mathcal{L}$. If $\mathfrak{X}$ is a metric space and $\mathcal{M}(\mathfrak{X})$ the space of all probability measures on the Borel field generated by $\mathfrak{X}$, $\mathbb{B}_{\mathfrak{X}}$, then the class Ω_o forms a closed semigroup on $\mathcal{M}(\mathfrak{X})$. In fact, with the topology of weak convergence and the multiplication defined by the convolution, $\mathcal{M}^{\omega,*}$ becomes a topological group. The infinitely divisible distribution structure of Banach and Hilbert space valued random variables is treated in Gikhman-Skorohod (1974). In the next section we shall briefly see some connection between infinitesimal random systems defined on finite and infinite dimensional function spaces. However, with slight modifications in terminology, the main results of that section such as theorems 4.2, 4.3, and 4.4. remain valid in Banach and Hilbert spaces, or indeed any other suitable extended space.

Now, let $X_1, X_2, \ldots$ be independent random variables such that for a suitable choice of $B_n > 0$ and A_n the sequence of distributions of, $Y_n = B_n^{-1} \sum_{i=1}^{n} X_i - A_n$, has a limit distribution F(x). This problem of limit sums is very old, since the beginning of the eighteenth century, but a considerable progress has been made during the second quarter of this century. Now, define S_m (m = 0, 1, 2, ...,) inductively as follows. Let S_o be the class of sequences $\{X_n\}$ of independent random variables generating convergent triangular arrays. Then S_o is the class of sequence $\{X_n\}$ of independent random variables such that for a suitable chosen random constants $A^*_n, n^{-1} \sum_{i=1}^{n} X_i - A^*_n$ has a limit distribution. Define S_m ($m = 1, 2, \ldots, \infty$) to be the class of all sequences $\{X_n\}$ such that $\{X_n\} \varepsilon S_o$ and for every positive real constant γ the triangular array $X_{nj} = X_{[\gamma n]+j}$ is equivalent to an array generated by a sequence from S_{m-1}, here $[\gamma n]$ denotes the integral part of the number γn. Let $S_\infty = \bigcap_{i=0}^{\infty} S_i$ and $L_\infty = \bigcap_{i=0}^{\infty} L_i$, where L_i is the set of all possible limit distributions of normal sums $n^{-1} \sum_{i=1}^{n} X_i - A^*_n$ where

$\{X_n\} \in S_i$ and $a_n \in \mathcal{R}$. We define L_{-1} to be the set of all probability measures on $\mathcal{R}$. Obviously, S_m form a contracting sequence of sets and hence L_m form a decreasing sequence of limiting distributions.

The case L_o was solved by Lévy, see Loève (1963, page 326), which is our class $\mathcal{S}_1$ containing all stable laws. The structure of L_2 has been discussed by Koroljuk-Zolotarev in 1961, where they established that in this case F(x) must be a convolution of two stable distributions. For $r > 2$ Zinger in 1965 showed that F(x) is a convolution of r stable distributions. Notice that $\mathcal{L}_\infty$ is the smallest class containing all stable distributions, that is the family $\mathcal{S}_2$. Furthermore, S_∞ as expected form the class of all slowly varying functions. Thus to summarise, we have the following.

Theorem 3.3. (i) $\mathcal{S}_3 \sqsubset \mathcal{S}_2 \sqsubseteq L_\infty \sqsubset \cdots \sqsubset L_1 \sqsubset L_o = \mathcal{S}_1 \sqsubset \mathcal{S}_o \sqsubset \mathcal{S}_{CLP} \sqsubset \tilde{\mathcal{S}}_\infty = \{F_\phi : \phi(0) = 1\}$, see corollary 2.1. (ii) A probability measure μ belongs to L_m ($m = 0, 1, \ldots, \infty$), if and only if, for every $\alpha \in (0,1)$, there exists a probability measure $\mu_\alpha \in L_{m-1}$ such that $\mu = \tilde{\mu}_\alpha * \mu_\alpha$, where $\tilde{\mu}_\alpha$ is defined by $\tilde{\mu}_\alpha(A) = \mu(\alpha^{-1}A) = \mu\{\alpha^{-1}x : x \in A\}$.

Notice that the restriction proposed by Urbanik (1973) is essentially a stability condition. Under these circumstances Urbanik was able to prove that a function ϕ is the characteristic function of a probability measure from L_∞, if and only if,

$$(3.9) \quad \log \phi(t) = iat - \int_{-2}^{2} \left[|t|^{|x|} (\cos \pi x/2 - it|t|^{-1} \sin \pi x/2) + itx \right] \frac{\mu(dx)}{1-|x|}$$

where a is a real constant, μ is a finite Borel measure on $(-2,0) \sqcup (0,2)$, and the integral is defined as its limiting value $(\pi/2)|t| + it \log |t| - it$ for $x = 1$, and $(\pi/2)|t| - it \log |t| + it$ when $x = -1$.

4. A UNIFIED EXTENDED ALGEBRAIC STRUCTURE APPROACH.

Let us recall the well known and very useful functional equation:

$$(4.1) \qquad f(s)f(t) = f(s+t), \; f(0) = 1; \quad s,t \geq 0,$$

which is satisfied by $f(t) = e^{\lambda t}$, λ any constant. This equation brings in mind the Poissonian and Compound Poissonian distributions on one hand; and the study of exponential functionals on infinite-dimensional function spaces, that is on the other hand, the semigroup theory of operators. In fact, we shall see that this ties up very neatly with the so called compound Poissonian semigroups, stochastic semigroups, Markovian semigroups, and the famous Hille-Yosida-Phillips theorem in functional analysis. Before we proceed further, we need some terminology. Let

C_B = the class of all bounded continuous functions,

C_∞ = the class of all continuous functions with finite limits $f(-\infty)$ and $f(+\infty)$,

C_o = the class of all continuous functions vanishing at $\pm\infty$,

C = the class of all continuous functions,

$C^\infty = \{f : f \in C_\infty$ and f has derivatives of all order in $C_\infty\}$

= the family of infinitely differentiable functions.

Clearly, $C \sqsupset C_B \sqsupset C_\infty \sqsupset C^\infty$, C_o is the closure of C_K, the class of continuous functions {f} vanishing outside a compact set K_f, with respect to uniform convergence.

A convolution semigroup $\{\mathcal{G}(t),\; t \geq 0\}$ is a class of operators associated with probability distributions $\{G_t\}$ on R and satisfying: $\mathcal{G}(s+t) = \mathcal{G}(s)\mathcal{G}(t)$. In fact, if

$$(4.2) \qquad G_s * G_t = G_{s+t} \text{ and } \mathcal{G}(t)u(x) = \int_{-\infty}^{\infty} u(x-y)\,dG_t(y),$$

then $\mathcal{G}(s+t) = \mathcal{G}(s)\mathcal{G}(t)$ is equivalent to $G_{s+t} = G_s * G_t$. Note that for $\{\mathcal{G}(t)\}$ the domain is C_o, and these operators are transition operators, that is, $0 \leq u \leq 1$ implies $0 \leq \mathcal{G}(t)\, u \leq 1$ and $\mathcal{G}(t)1 = I$, where I is the identity operator. The convolution semigroup $\mathcal{G}(t)$ is said to be continuous if $\mathcal{G}(t) \to I$ as $t \to 0+$. In this case we write $\mathcal{G}(0) = I$. Finally, an operator $T: C^\infty \to C^\infty$ is said to generate the convolution semigroup $\{\mathcal{G}(t)\}$ if as $t \to 0+$, $t^{-1}(\mathcal{G}(t) - I) \to T$. Equivalently, one can say T is the generator.

Theorem 4.1. If $\{\mathcal{G}(t)\}$ and $\{\mathcal{G}^*(t)\}$ are two continuous semigroups having the same generator T, then $\mathcal{G}(t) = \mathcal{G}^*(t)$ for all $t \geq 0$.

Proof. By the Hille-Yosida-Phillips theorem there exists a real number λ^* such that $L(\lambda,T) = (\lambda I-T)^{-1}$ exists and is bounded for every $\lambda > \lambda^*$. So, we find that

$$L(\lambda,T) = \int_0^\infty e^{-i\lambda t}\, T_G(t)u\, dt = \int_0^\infty e^{-i\lambda t}\, T_{G*}(t)u\, dt.$$

Hence, for any f in the set of continuous linear functionals in the space of definition, $f[T_G(t)u]$ and $f[T_{G*}(t)u]$ have the same Laplace-Stieltjes transforms, and since both are $O(e^{at})$ for some finite a coincide. Since f is arbitrary the result is proved.

Now, we are in a position to state and prove the Basic Structure Theorem which unifies and inter-relates the various concepts involved. In what follows $Q_t = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} G^{k*}$, with an arbitrary probability distribution G, and $\lambda > 0$, defines a compound Poisson process. Clearly, the family of such distributions forms a semigroup since it satisfies $Q_{s+t} = Q_s * Q_t$, s, $t > 0$. An alternative version of the above is the following. Let $Y_1, Y_2, \ldots$ be independent random variables with distribution G, and suppose N(t) is a Poisson process with parameter t and independent of Y_j. Then Q_t is the distribution of $\sum_{j=1}^{N(t)} Y_j$. The compound Poisson processes have many applications such as traffic jams, fishing, risk theory and insurance, water or oil reservoir contents, customers arriving at a server, signal detection, and so on. Notice that in the definitions of infinite divisibility and compound Poisson processes the assumption of the same distribution is no restriction, since there exists an appropriate group which generates the whole family of distributions involved.

Theorem (Basic Structure) 4.2. The following families of probability distributions or corresponding measures are identical.

(a) M_1 = the class of all infinitely divisible distributions.

(b) M_2 = limits of sequences of infinitely divisible distributions.

(c) M_3 = limit distributions of row sums in infinite random systems $\{X_{jn_j}\}$ where the variables X_{jn_j} of the nth row have a common distribution.

(d) M_4 = the distributions of increments in processes with stationary independent increments, that is, distributions associated with continuous convolution semigroups.

(e) M_5 = limit of sequences of compound Poisson distributions, that is, distributions generated by compound-Poissonian semigroups.

Let $M_1 \equiv M_2 \equiv M_3 \equiv M_4 \equiv M_5 \equiv \Omega$. Thus, for a given infinitesimal random system there exists a unique (upto an equivalence class) generator T which generates the family Ω via the appropriate convolution semigroup, and conversely.

Proof. For each n let G_n be an infinitely divisible distribution. By definition G_n is the distribution of the sum of n independent identically distributed random variables, and so the sequence $\{G_n\}$ generates an infinitesimal system in M_3. Thus, $M_2 \subseteq M_3$.

Next, we show $M_3 \subseteq M_4$. Let T_n be the operator induced by the distribution $G_{X_{k,n}}(\cdot)-b_n = F_n(x+b_n)$. To show that the distribution of $\sum_{j=1}^{n_k} X_{jn_j} - n_k b_{n_k}$ tends to the distribution G_1 associated with $\mathcal{G}(I)$ it is sufficient to show [see Feller (1971) pages 311-312] that as n runs through $\{n_k\}$, $n(\mathcal{G}_n - I) \to T$, where $Tu(x) = \int_{-\infty}^{\infty} y^{-2}[u(x-y) - u(x) + T_s^*(y)\, u'(x)]d\mu(y)$ with μ an appropriate canonical measure. Here, the truncation function T_s^* is the continuous monotone function such that $T_s^*(x) = x$ when $|x| \leqslant s$ and $T_s^*(x) = \pm s$ when $|x| \geqslant s$, where $s > 0$ is arbitrary but fixed. The uniqueness of the semigroup containing G_1 shows that the limit relations are valid for an arbitrary approach $n \to \infty$. This proves that $M_3 \subseteq M_4$.

Now, we show that $M_4 \subseteq M_5$. Since the operator $T_h = \frac{\mathcal{G}(h)-I}{h}$ for fixed $h > 0$ generates a compound-Poissonian semigroup of operators $\mathcal{G}_h(t)$, as $h \to 0+, T_h \to T$, and hence $\mathcal{G}_h(t) \to \mathcal{G}(t)$. Thus, G_t is the limit of compound-Poisson distribution, and consequently $M_4 \subseteq M_5$. On the other hand, $M_5 \subseteq M_2$ trivially. Finally, $M_1 \subseteq M_2$ and $M_1 \supseteq M_4$. This completes the proof of the theorem.

If $\{\phi_n, n \geqslant 1\}$ is a sequence of infinitely divisible characteristic functions converging everywhere to characteristic function ϕ, then ϕ is infinitely divisible. By employing this fact we can construct a wide class of characteristic functions in Ω as follows. For each $\lambda > 0$ and real u, consider the function $h(t;\lambda,u) = \exp(\lambda \exp(iut - 1))$, which is an infinitely divisible characteristic function. Clearly, $\exp[\sum_{j=1}^{k} \lambda_j(\exp(iu_j t-1))]$ is also infinitely divisible. Furthermore, if H is a bounded increasing function, $\phi^*(t) = \exp[\int_{-\infty}^{\infty}(\exp\,(itu)-1)dH(u)]$ is an infinitely divisible characteristic function. Thus, for every $\phi \in \Omega$, there exists a double array of real numbers $\{\lambda_{n_j}, u_{n_j}\}$, $1 \leqslant j \leqslant k_n$, $1 \leqslant n$, where $\lambda_j > 0$, such that

$$\phi^*(t) = \lim_{n\to\infty} \prod_{j=1}^{k_n} h(t;\lambda_{n_j}, u_{n_j}).$$

The converse is also true. Notice that the above discussion leads naturally to the well-known result due to De Finetti which states that ϕ is in Ω,if and only if,$\phi(t) = \lim_{n\to\infty} \exp(\lambda_n(\phi_n^*-1))$ where λ_n is a real number,ϕ_n^* a characteristic function.

A measure μ on (R^1,B^1) with $\mu(R^1) \leqslant 1$ is called a defective probability measure (ipm). The sequence $\{\mu_n, n \geqslant 1\}$ of dpm's is said to converge vaguely to a dmp μ (written $\mu_n \xrightarrow{v} \mu$), if and only if, there exists a dense subset D of R^1 such that for every a in D, b in D, a < b: $\mu_n((a,b]) \to \mu((a,b])$. The general criterion for vague convergence is as follows. If $\{\mu_n\}$ and μ are pm's then $\mu_n \xrightarrow{v} \mu$, if and only if, one of the two following conditions is satisfied:

(a) for every f in $\tilde{L}$: $\underline{\lim}_n \int f(x)d\mu_n(x) \geqslant \int f(x)d\mu(x)$;

(b) for every g in $\tilde{U}$: $\overline{\lim}_n \int g(x)d\mu_n(x) \leqslant \int g(x)d\mu(x)$;

where $\tilde{L}$ = {bounded lower semicontinuous functions, that is, for every

$$x \text{ in } R^1:\quad f(x) \leqslant \underline{\lim}_{\substack{y\to x\\ y\neq x}} f(y)\},$$

and $\tilde{U}$ = {bounded upper semi-continuous functions}.

Thus, in view of the above considerations, the class Ω coincides with the closure, with respect to the vague convergence of convolutions of a finite number of Poisson distribution functions of the form $h(t;\lambda,u)$ above.

Recall that under Weak Law of Large Numbers (WLLN) and Central Limit Theorems (CLT) sums of independent random variables converge in distribution to a degenerate distribution with characteristic function $\exp(\lambda it)$ and normal distribution with characteristic function $\exp(ait-bt^2)$. Note that both of these characteristic functions are exponentials of polynomials of the first and second degree in (it). Other interesting cases of limiting distributions are: Poisson with characteristic function $\exp(\lambda e^{it}-1)$, $\lambda > 0$; and symmetric stable distributions of exponent α with characteristic function $\exp(-\lambda|t|^\alpha)$, $0 < \alpha < 2$, $\lambda > 0$. For $\alpha = 1$ (2) we have the Cauchy (normal) distribution. The above considerations lead us to the following unified structure of such distributions. But first we need the theorem below, which can also be deduced from the theorem 2.1.

Theorem 4.3. Let a complex-valued function ϕ of real variable t be given. Assume $\phi(0) = 1$ and that ϕ is continuous in $(-\infty,\infty)$ and does not vanish in the interval. Then there exists a unique single-valued function ψ of t in $(-\infty,\infty)$ with $\psi(0) = 0$ that is continuous and satisfies $\phi(t) = e^{\psi(t)}$. The function ψ is called the distinguished logarithm, and $e^{\psi(t)/n}$ the distinguished nth root of ϕ.

Proof. It follows by employing slightly modified arguments as in Chung (1974; Theorem 7.6.2, pages 241-242), or can be obtained by using part (ii) of the theorem 2.1.

Thus, there is a one-to-one correspondence among an infinitely divisible distribution F, ϕ_F, and ψ_ϕ. Furthermore, if one is considering the relationship between infinitely divisible probability distributions and helical varieties in Hilbert spaces, see Masani (1973), then ψ is uniquely determined by a suitable kernel $\mathcal{K}_\psi$. There is a slight departure from uniqueness when ψ is complex-valued. For a helical variety (or helix), h, the corresponding chordal covariance kernel $\mathcal{K}^*_{\tilde{h}}$ is translation-invariance. This fact, under appropriate spaces, connects helical varieties with random processes or random fields with wide-sense stationary increments. Consequently, there is correspondence, although not one-one, between ϕ_ψ and h. Notice that the very idea of helix comes from Brownian motion, and is examplified by all random processes with stationary increments. Of course, only in cases dominated by normality do strict-sense and wide-sense concepts merge. In general, the strict-sense concepts such as stationarity, Markovian, martingale property etc., belong to the theory of probability, and the wide-sense concepts are in the domain of the theory of Hilbert and Banach spaces, as pointed out by Masani (1973).

Now, let $\breve{\mathfrak{H}}$ be a maximal group of measure-preserving transformations, which generates classes {h} or {ψ} discussed above. For a distribution F, whether in Ω or not, if there exists at least one h in $\breve{\mathfrak{H}}$ such that $\tilde{F} = F_{\tilde{h}}$ is in Ω, then we call such a distribution F an $\breve{\mathfrak{H}}$-potential generator. Clearly, the group $\breve{\mathfrak{H}}$ excludes some distributions. For example, for no h, $F \equiv U[-a,a]$ where $U_{[-a,a]}$ is a uniform distribution on $[-a,a]$ which is not infinitely divisible since its characteristic function is $\frac{\sin at}{at}$ which vanishes for some real t, although it has many divisors.

Theorem (Extended Equivalent Class Structure) 4.4. Let Ω, $\mathcal{G}$, and $\breve{\mathfrak{H}}$ be as defined above. Then $\breve{\mathfrak{H}}$ generates Ω, that is, for every G in Ω, there exists at least one $\tilde{F} = F_{\tilde{h}}$ such that $\mathcal{G}_G \equiv \mathcal{G}_{\tilde{F}}$, $h \in \breve{\mathfrak{H}}$.

Proof. By the very structure of the group $\breve{\mathfrak{H}}$, there exists a continuous homomorphism; $\tilde{\phi}: \breve{\mathfrak{H}} \to \mathcal{G}$. Therefore, by the Basic Structure Theorem for any distribution G generated by $\mathcal{G}$, there exists $F_{\tilde{\phi}}$ which is an $\breve{\mathfrak{H}}$-potential generator. Hence, to the family {Q(t)} there corresponds a class {$\tilde{Q}(t;\tilde{\phi},h)$} which generates an appropriate semigroup. Consequently, from $\mathcal{G}_{\tilde{Q}}$ one can construct the whole class of infinitely divisible distributions. This proves $\Omega(\breve{\mathfrak{H}};\ \{F\}) \equiv \Omega \equiv \Omega(\mathcal{G})$, where $\Omega(\mathcal{G})$ is the family of all infinitely divisible distributions generated by $\mathcal{G}$. Notice that the family {$\tilde{Q}$} has some resemblance to the class h(t; .,.) discussed after the Basic Structure Theorem.

Corollary 4.1. (i) If $F \in \Omega$ and $F * G \underset{n}{=} H \in \Omega$, then $G \in \Omega$. (ii) If F_j $(j = 1, 2, \ldots, n < \infty)$ is in Ω, then $\sum_{j=1}^{n} w_j F_j \in \Omega$, where $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. (iii) If ϕ_n converges weakly to ϕ, ϕ_n an infinitely divisible characteristic function, then by virtue of the theorem 4.3, ϕ is an infinitely divisible.

It was pointed out earlier that the set of all probability measures $\mathfrak{M}$ with convolution as multiplication and weak (or vague) convergence forms a topological group $\mathfrak{M}^{*,w}$. Let Σ be a group and denote by $\mathbb{B}_\Sigma$ its Borel field and $\mathfrak{M}(\Sigma)$ as the space of all measures defined on $\mathbb{B}_\Sigma$. Some notable Indian and Russian probabilists have introduced an extended definition of infinite divisibility. That is, μ is infinitely divisible if there exists a single element $\sigma \in \Sigma$ such that $\mu = \mu_n^{n*} * \sigma$, $\mu_n \in \mathfrak{M}(\Sigma)$. A measure $\mu \in \mathfrak{M}(\Sigma)$ is called idempotent if $\mu * \mu = \mu$. Let Σ be a complete separable metric space and μ an idempotent measure on Σ. Then it is well known that there exists a compact subgroup $\Sigma^* \subseteq \Sigma$ such that μ is the normalized Haar measure of Σ^*. A Haar measure is a left (or right) invariant Borel measure which is not identically zero. Furthermore, a Haar measure is not unique, that is, for each $a \in R_+^1$, $a\mu$ is a Haar measure if μ is. Clearly, every locally compact topological group has a Haar measure. Let ϕ_μ be the characteristic function of an infinitely divisible measure μ. If $\phi(\bar{u}) = 0$ for some character $\bar{u}$ then μ has an idempotent factor. So, the normalized Haar measure of a compact subgroup Σ^* is an example of an infinitely divisible distribution. This fact is a consequence of the result that such a measure is idempotent. Thus, if we have a locally compact (vague) topological group, then it has a Haar measure, and this measure should generate other infinitely divisible distributions. Notice the relationship between this observation and our extended equivalent class algebraic structure theorem.

In considering the structure of stochastic independence, Bell (1958) has given interrelationship among various concepts of independence: (a) sigma-independence; (b) almost-sigma-independence; (c) stochastic-independence; and (d) quasi-sigma-independence. The solution proposed by Bell is as follows: no two of the conditions are equivalent, and $a \to b \to c \to d$. The result above motivates one to seek classes larger than the class Ω which, in some appropriate sense, exhibit essentially a similar structure as the class of infinitely divisible distributions. To the best of the authors' knowledge no results on these lines exist in the literature. By definition, F is infinitely divisible, if and only if, for each n it can be represented as the distribution of the sum $S_n = \sum_{j=1}^{n} X_{jn}$ of n independent random variables with a common distribution F_n. In view of this fact, we propose that 'independence' should be replaced by some 'weaker form of independence' or some 'tractable form of dependence', and then some 'weaker form of infinite divisibility' behaviour should be studied. We hope to take up these aspects in a further paper. We may recall that the class Ω belongs to that of limit laws of the central limit problem. If we denote the later family by Ω_{CLP}, then in the above we are talking about a family Ω' or $\bar{\Omega}$ such that $\Omega \subset \Omega' \subset \Omega_{CLP} \subset \bar{\Omega}$. We may also notice that similar results as theorems 4.2 and 4.4 hold even in nonstationary processes with independent increments, provided a suitable continuity condition is imposed to assure the existence of an infinitesimal random system.

5. ON THE UNIMODALITY OF CLASSES Ω_1, Ω_2 AND Ω_3:

The unimodality of a process is one of the nice and useful behaviour properties which help obtain better statistical inferences. From the definition we have noticed that $F'(x)$ is continuous everywhere with a unique finite maximum either a single point or a connected interval. It is known, Gnedenko-Kolmogorov (1954, p. 160), that a distribution $F(x)$ is unimodal with vertex at $x = 0$, if and only if, its characteristic function $\phi(t)$ can be represented in the form:
$\phi(t) = t^{-1}\int \phi^*(s)ds$, $-\infty < t < \infty$, where $\phi^*(t)$ is some characteristic function. For

example, $\phi(t) = (1+|t|^{\alpha})^{-1}$, $0 < \alpha \leqslant 2$ is the characteristic function of a unimodal distribution. The structure of unimodality in the literature can be summed up as follows.

Lemma 5.1. (i) If F and G are (symmetric unimodal) distributions, then so is $F * G$. (ii) All stable distributions with characteristic functions given by $\exp(-|t|^{\alpha})$ are unimodal. Consequently all members of the class Ω_3 are unimodal. (iii) All distributions of the class Ω_2, that is, all stable distributions are unimodal.

Definition 5.1. A distribution $\Gamma(x)$ is said to be strongly unimodal, if and only if, the convolution of F with any unimodal distribution is unimodal. [For example, the normal and Wishart distributions are strongly unimodal].

Lemma 5.2. A distribution from the class Ω_1 is unimodal if at least one of the following conditions is valid: (i) F is symmetric; (ii) F is stable; (iii) F is symmetric stable; (iv) either L^- or L^+ is zero; (v) $F = N * G$, where N is a normal distribution and G any unimodal distribution, or equivalently $F = G_s * G^*$, where G_s is a strongly unimodal distribution and G^* an arbitrary unimodal distribution.

In the Russian edition of Gnedenko-Kolmogorov book, which appeared in 1949, there appeared a theorem due to Gnedenko, stating that every L-function is unimodal. This result depended on an incorrect theorem due to Lapin in his 1947 thesis, which states that the convolution of two unimodal distributions with vertex at zero is unimodal with vertex at zero. Chung (1953) gave two counter examples and presented a correct version of Lapin's result. Ibragimov (1957) gave some examples of distributions in Ω_1 which were not unimodal, but Sun (1967) proved the unimodality of Ibragimov's examples. With these historical remarks, in the sequel we give some new results.

Theorem 5.1. Let $F(x) = F_1(x) * F_2(x)$ be an $\mathcal{L}$-function with the Lévy spectral functions L^- and L^+ such that the corresponding distribution functions F_1 and F_2, respectively, satisfy

$$(5.1) \qquad V(x) = \tau_1 * F_2 + \tau_2 * F_1 - F_1 * F_2 + s(F_1' * F_2),$$

V is a distribution, where $\tau_i(x) = F_i(x) * F_i'(x)$ $(i = 1, 2)$ and s is the vertex of $F_1 * F_2$, $V(x)$ may be either left or right continuous, and $F_i'(x)$ is the left derivative. Then F_1 and F_2 are unimodal and consequently F is unimodal. Notice that in fact $F = N(a,\sigma^2) * F_1 * F_2$ where N has trivial Lévy spectral measures.

Proof. Clearly $L^-(x)$ and $L^+(x)$ with respect to $F(x)$ are absolutely continuous. If we let $xL^{-\prime}(x) = \delta(x)$ and $xL^{+\prime}(x) = \delta^*(x)$, then the Lévy canonical representation can be written in terms of $\delta(x)$ and $\delta^*(x)$. The Lévy spectral functions $L^-(x)$ and $L^+(x)$ have right and left derivatives for every x and the functions $\delta(x)$ and $\delta^*(x)$ are nonincreasing, here $L^{-\prime}(x)$ and $L^{+\prime}$ denote either right or left derivative, possibly different ones at different points.

Now, $\phi(t)$ can be written as $\phi(t) = \phi_o(t)\, \phi_1(t)\, \phi_2(t)$, where ϕ_1 and ϕ_2 are characteristic functions of F_1 and F_2, and ϕ_o is the characteristic function of $N(a,\sigma^2)$. Since

$$\phi_2(t) = \exp\Big[\int_o^\infty (e^{itx} - 1 - \frac{itx}{1+x^2}\, \frac{\delta^*(x)}{x}\, dx\Big],$$

it is possible to construct a sequence of nonincreasing step functions $\{\delta_j^*(x)\}$ such that $0 \leqslant \delta_1^*(x) \leqslant \dots \leqslant \delta_j^*(x) \leqslant \dots$, and $\delta_n^*(x) \to \delta^*(x)$ as $n \to \infty$ for $x > 0$. For each n let $F_{2n}(x)$ denote the $\mathcal{L}$-function with characteristic function

$$\phi_{2n}(t) = \exp\Big[\int_o^\infty (e^{itx} - 1 - \frac{itx}{1+x^2}\, \frac{\delta_n^*(x)}{x}\, dx\Big].$$

Let $G_{2n}(x)$ and $G_2(x)$ be the Lévy-Khintchine spectral functions of $F_{2n}(x)$ and $F_2(x)$, respectively, where $G(x)$ is as defined by (3.6). It is easy to see that $G_{2n}(x) \to G_2(x)$ as $n \to \infty$ for all values of x. Then from the fact that almost sure convergence in probability, we get that $F_{2n}(x) \to F_2(x)$ as $n \to \infty$, and that F_2 is unimodal. By a similar argument one can prove that $F_1(x)$ is unimodal. If any of F_1 or F_2 is with vertex $s_i \neq 0$ $(i = 1, 2)$, we can easily shift the vertex to the origin. Because of the validity of (5.1), $F_1 * F_2$ is unimodal at s and $\phi^*_{1,2}(t) = \phi_1(t)\ \phi_2(t)$. Since $\phi_o(t) = \exp(iat - t^2\sigma^2/2)$ is the characteristic function of $N(a,\sigma^2)$ which is strongly unimodal, by the definition of strong unimodality and the fact that a continuous distribution is strongly unimodal, if and only if, the logarithm of its density is concave, we can see that $\phi(t) = \phi_o(t) \times \phi^*_{1,2}(t) = \phi_o(t)\ \phi_1(t)\ \phi_2(t)$ is a characteristic function of a unimodal distribution. This completes the proof of the theorem.

Now, one can compare the following corollaries with the main results of Wolfe(1971).

Corollary 5.1. (i) If $L^-(x) = 0$ or $L^+(x) = 0$ in the theorem above, then the $\mathcal{L}$-function with the characteristic function of the Lévy canonical representation is unimodal. (ii) Any $\mathcal{L}$-function which is unimodal is a convolution of at the most three distinct unimodal distributions. The convolution of two unimodal infinitely divisible distributions need not be unimodal.

6. SOME COMMENTS:

Barndorff-Nielsen (1976) has defined the Universality concept in connection with plausibility inference. It turns out that, under some mild regularity conditions, for the regular exponential family the concepts of strong unimodality and universality are essentially the same. LeCam (1960, 1974) in constructing asymptotically sufficient estimates, uses the following approximating functions: $L(t,x;\alpha,\beta) = -\alpha^2\beta^2[\exp(\beta(x-t))-1-\beta(x-t)]$ for $\alpha > 0$, $\beta \neq 0$, β in $(-\infty,\infty)$, and $L(.) = -\alpha^2(x-t)^2/2$ for $\alpha > 0$, $\beta = 0$. Comparing these functions with the integrand kernels of the basic representation in section 3, one realizes an important application of the class Ω_o and its subfamilies. For more details see Ahmad-Al-Mutair (1976). Let the summand S_n from an i.i.d. sequence with any distribution F, have distribution F_n. Then, Ibragimov-Linnik (1971, p. 267) $\exists$ a sequence $\{F_n{}^o\}$ from Ω s.t. $\sup_x |F_n{}^o(x)-F_n(x)| \leq cn^{-1/3}$, where c is an absolute constant. Notice the implication of this global approximation to the situation above.

In connection with multivariate unimodality there are several definitions such as monotone unimodal (MU), central convex (CCU), and convex unimodal (CU), see Dharmadhikari-Jogdeo (1976) and the references cited there. Here we prove a conjecture of Sherman and give an affirmative answer to two of the Dharmadhikari-Jogdeo open questions. This depends on the lemma below. Due to lack of space our discussion is brief.

Lemma 6.1. Let E denote a closed convex set. Then for any C in E and every non-negative t, the set of all vectors X such that C + tX is in E is closed convex cone independent of C.

Proof. Without loss of generality we may take an element $y \in E$ instead of the set C and a point x instead of the vector X. Call ray, for a cone which is a proper subset of a line. Now, one can easily see that the sets of elements x such that $y + tx \in E$ is a convex cone. For non-negative t the set of all x such that $y + tx \in E$ is a closed set, but the above cone is the intersection of these closed sets. Therefore, it is closed. To prove the independent part, take any element z of E. Suppose $y + tx \in E$ for $t > 0$. The set E is closed convex set, and it contains the point z and the ray with origin y and direction x. The E contains the closed convex hull of the set formed by z and this ray. Whereas this ray may be said contains the ray with origin z and direction x. Hence the considered cone is independent of y. This proves the lemma.

Let $\tilde{L}$ be a linear space, $\tilde{T}$ be a topological space, $\tilde{T}\tilde{L}$ be a topological linear space which can be obtained by defining the topology on subsets of the linear space $\tilde{L}$. A convex set of $\tilde{T}\tilde{L}$ which has an interior point is called a convex body.

Now, MU and CCU are symmetric, and for simplicity we shall take the symmertization to be around the origin. In every MU defined above we can take f(C+tX) to be bounded. Now, recall Dharmadhikari and Jogdeo (1976) corollary 3.1, which states that a distribution F in R^k is MU, if and only if, the defining property holds for all symmetric compact convex bodies C in R^k. From the above lemma, previous discussion, the corollary, and the space $\tilde{TL}$ structure, one concludes that MU implies CCU and both the families MU and CCU are closed under convolutions.

Finally, to conclude we mention some open problems. Are all univariate and multivariate $\mathcal{L}$-distributions unimodal, in some suitable sense? What is the essential structure (other than what is already known) of strong unimodality, stability, symmetry and universality in the multivariate case? Are multivariate $\mathcal{L}$-distributions absolutely continuous?

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