Accepted for publ.
Statistica Neerlandica, 87

ON DISCRETE α - UNIMODALITY

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Key words & Phrases: α - unimodality, Characteristic function, Mixture, Convolution, Non-lattice distribution.

ABSTRACT

A definition of α – unimodality for integer-valued random variables is introduced. Some characterization results in terms of probability mass functions and characteristic functions are established. Properties of generalized discrete unimodality are investigated. Further, by use of limiting properties, discrete α- unimodality is shown to imply the concept of α - unimodality for the real-valued random variables due to Olshen and Savage (1970).

1. INTRODUCTION

A real random variable (r.v.) X is said to be unimodal about m if its distribution function F(x) is convex for $x \leqslant m$ and concave for $x \geqslant m$. Khintchine (1938) and Isii (1958) have independently shown that the real r.v. is unimodal about zero, if and only if it is equal in distribution to the product of two independent real r.vs., YU, say, where U is uniform on [0,1]. Olshen and Savage (1970) have introduced the concept of α - unimodality. They call the real-valued r.v. X α- unimodal if and only if it is equal in distribution to $YU^{1/\alpha}$ with U and Y as above.

AMS 1979 Subject Classification: 60E05, 62E10:

The concept of α - unimodality is related to coverage problems, Olshen and Savage (1970). Estimation of unimodal densities and estimation of density function by mixtures of unimodal kernels have been investigated by many authors, see Wegman (1972) and Silverman (1981) and references therein. Such applications might be investigated under the assumption of α - unimodality for values of α other than one.

Discrete unimodality has been studied by many authors, such as Keilson and Gerber (1971), Medgyessy (1972), Dharmadhikari and Jogdeo (1977), Abouammoh and Mashhour (1981, 1983), and Bertin and Theoderescu (1981).

In this section, α - unimodality for integer-valued r.vs. is introduced. In section 2, some characterization results in terms of probability mass functions (p.m.fs.) and characteristic functions (ch.fs.) are established. In section 3, some properties and examples of α - unimodality are given. Finally in section 4, it is shown that the observation of Olshen and Savage (1970) about expressing the α - unimodal real r.v. as a product of two independent r.vs. can be established as a limiting result of discrete α- unimodality.

The distribution p_n, $n \in \mathbb{Z} = [\ \ldots, -2,-1,0,1,2, \ldots\ ]$ is said to be unimodal if there exists at least one integer m such that

$$\begin{array}{ll} p_n \geqslant p_{n-1} & n \leqslant m \\ p_n \geqslant p_{n+1} & n \geqslant m, \end{array} \qquad (1.1)$$

The point m is called a mode of the distribution. The mode need not be unique. Many distributions such as the degenerate, the uniform, the Poisson, the geometric and the binomial are unimodal. The definition of unimodality given by (1.1) can be generalized as follows. A distribution p_n, $n \in \mathbb{Z}$ is said to be α- unimodal about zero if

$$\begin{aligned} (\alpha - n)\, p_n &\geqslant (1-n)\, p_{n-1} \qquad n \leqslant 0 \\ n\, p_n &\geqslant (n+1-\alpha)\, p_{n+1} \qquad n > 0 \end{aligned} \qquad (1.2)$$

where $\alpha \geqslant 1$. For $\alpha = 1$, the α- unimodality reduces to ordinary unimodality given by (1.1)

2. CHARACTERIZATION

Here, we give some characterization results of discrete unimodality in terms of p.m.fs. and ch. fs. We shall use the following notation. If p_n and q_n, $n \in \mathbb{Z}$ are probability distributions on Z, then their distribution functions are denoted by $P_n = \sum_{-\infty}^{n} p_k$ and $Q_n = \sum_{-\infty}^{n} q_k$.

Theorem 2.1 The distribution p_n, $n \in Z$, with finite first moment is α- unimodal about zero if, and only if Q_n defined by

$$Q_n = P_n - \alpha^{-1} n p_n, \qquad n \in \mathbb{Z} \qquad (2.1)$$

is a discrete distribution function.

Proof : Let P_n, $n \in \mathbb{Z}$ be α- unimodal distribution and define Q_n by (2.1). Next, consider (2.1) with n replaced by n-1, i.e.

$$Q_{n-1} = P_{n-1} - \alpha^{-1}(n-1)\, p_{n-1} \qquad (2.2)$$

The difference of (2.1) and (2.2) gives

$$q_n = Q_n - Q_{n-1} = (1 - \alpha^{-1} n)\, p_n + \alpha^{-1}(n-1)\, p_{n-1}, \qquad (2.3)$$

or equivalently.

$$\alpha q_n = (\alpha - n)\, p_n + (n-1)\, p_{n-1} \tag{2.4}$$

Summing over n for $n \in \mathbb{Z}$, we get $\Sigma\, q_n = 1$. This means that q_n, $n \in Z$ is a discrete distribution.

The sufficient part, that is if Q_n, $n \in \mathbb{Z}$ given by (2.1) is a distribution function is established by showing that (2.4) implies relations (1.2). This completes the proof.

Next we give the above characterization in terms of ch.fs.

Theorem 2.2 A necessary and sufficient condition for a distribution function P_n, $n \in \mathbb{Z}$ with ch.f $p(t)$ and finite first moment, to be α- unimodal about zero is that $q(t)$ defined by

$$q(t) = p(t) + \frac{i}{\alpha}(1 - e^{-it})\, p'(t) ,$$

where $p'(t) = \frac{d}{dt} p(t)$, is the ch.f. of a discrete distribution q_n, $n \in \mathbb{Z}$.

Proof: Let P_n, $n \in \mathbb{Z}$ be α - unimodal distribution then (2.4) is true for some p.m.f. q_n, $n \in \mathbb{Z}$. Multiplying both sides by e^{int} and summing over $n \in Z$ gives

$$\alpha q(t) = \alpha p(t) + i(1 - e^{it})\, p'(t) \tag{2.5}$$

The converse of the theorem is proved in a similar manner.

The next result gives an integral representation for α- unimodal ch.fs.

Theorem 2.3 A ch.f. $p(t)$ with a finite first moment is the ch.f. of α- unimodal distribution p_n, $n \in \mathbb{Z}$ about zero if, and only if it can be represented as

$$p(t) = \alpha i (1 - e^{-it})^{-\alpha} \int_0^t e^{-i\alpha u} (e^{iu} - 1)^{\alpha - 1} q(u)\, du , \tag{2.6}$$

where $q(u)$ is the ch.f. of a discrete distribution Q_n, $n \epsilon \mathbb{Z}$.

Proof: By virtue of Theorem 2.2, $p(t)$ is the ch.f. of an α- unimodal discrete distribution if, and only if $p(0)=1$ and $p(t)$ satisfies (2.5).

This is a first order linear differential equation whose solution is given by (2.6).

Conversely, differentiating both sides of (2.6) with respect to t implies the differential equation (2.5) of Theorem (2.3). Hence the theorem.

The Theorems 2.1, 2.2 and 2.3 can be restated for the case when mode m may not be zero: P_n is α—unimodal about m if p_{n-m} is α- unimodal about zero.

It is noticed that the definition of α - unimodality can be generalized for $\alpha > 0$, but in this case we need an additional restriction, namely the distribution q_n defined by 2.3 has to satisfy $q_n \geqslant 0$ for $n = 1$.

3. PROPERTIES AND AN EXAMPLE

Definition (1.2) of α — unimodality immediately gives the following:

Theorem 3.1: If p_n, $n \epsilon \mathbb{Z}$ is α- unimodal and if $\beta > \alpha$, then it is β—unimodal.

This implies that many discrete distributions such as the Poisson, binomial, geometric, negative binomial and uniform, are α - unimodal for $\alpha \geqslant 1$.

From (2.1) it immediately follows that α - unimodality is preserved under convergence to a weak limit.

Theorem 3.2 Let $\{p_n^{\nu}\}$ be a sequence of α - unimodal distributions with modes m_{ν} , $\nu = 1, 2, \ldots$ and $n \epsilon \mathbb{Z}$. If $\{p_n^{\nu}\}$ converges weakly to some discrete distribution p_n, $n \epsilon \mathbb{Z}$ then p_n is an α—unimodal about m , where $m = \operatorname{Lim}_{\nu} m_{\nu}$.

It may happen that a mixture, of an α_1 -unimodal and an α_2 - unimodal distribution with the same mode but with $\alpha_1 \neq \alpha_2$ is not α- unimodal for any α .

The following theorem shows the preservation of two α- unimodal distributions under mixing for a fixed α . It follows from Theorem 3.2 that any mixture of α- unimodal distributions is α- unimodal. The proof is immediate from (2.1).

Theorem 3.4 Let p_n and q_n, $n \in \mathbb{Z}$ be α- unimodal distributions about some point m, then their mixture $r_n = \lambda p_n + (1-\lambda) q_n$, $n \in \mathbb{Z}$, and $0 \leq \lambda \leq 1$ is α- unimodal about m.

The following example shows that the generalized α- unimodality is not preserved in general under convolution. In fact, p_n and q_n may both be α - unimodal, and $p_n * q_n$ may not be α-unimodal for any α .

Example 3.5: Consider the distribution functions P_n, and Q_n, as given below. Let $H_n = P_n * Q_n$, i.e. the convolution of P_n and Q_n. Then for

n	<2	2	3	4	≥ 5
P_n	0	.125	.417	.611	1
Q_n	0	.500	.833	.888	1

The distribution H_n, $n \in Z$ is given by

n	< 4	4	5	6	7	8	9	≥ 10
H_n	0	.125	.292	.458	.745	.904	.948	1

It can be shown that the distribution functions P_n and Q_n, n = 2, 3, ... , 5 are 3/2 - unimodal, P_n has mode at n=3, whereas the distribution function H_n, n = 4,5, ... , 10 is not unimodal for any α.

4. LIMITING RESULT

Considering p_n as a distribution on $\lambda\mathbb{Z}$ rather than on $\mathbb{Z}$, see Feller (1971, p.136), by theorem 2.1 and by letting $\lambda \to 0$ we are led to the following definition.

<u>Definition 4.1</u>: A differentiable d.f $F(x)$, $x \in R$ is α- unimodal about zero, if and only if G defined by

$$G(x) = F(x) - \alpha^{-1} x F'(x) , \qquad (4.1)$$

where F' is the derivative of F, is a distribution function.

For $\alpha = 1$, (4.1) gives the definition of unimodality, see for example Gnedenko and Kolmogorov (1954, p.157).

Finally, we give the following result which shows that Definition 4.1 and the definition of α-unimodality given by Olshen and Savage (1970) are equivalent.

<u>Theorem 4.2:</u> A r.v. X is α- unimodal about 0, if and only if there exist two independent r.vs, U uniform on $[0,1]$ and Y such that $X \overset{d}{=} U^{1/\alpha} Y$, where $\overset{d}{=}$ means equals in distribution.

<u>Proof</u>: Let X be a r.v. with d.f. $F(x)$ and equal in distribution to $U^{1/\alpha} Y$ as above, and let H be the d.f. of Y. Then

$$F(x) = p(U^{1/\alpha} Y \leqslant x) = \int_0^1 H(X U^{-1/\alpha}) \, du ,$$

and one easily verifies that G as defined in (4.1) equals H. The converse is proved by taking $H = G$.

ACKNOWLEDGEMENT

The author is grateful to Dr. D.N. Shanbhag of Sheffield University for some useful discussions regarding this paper and to a referee for several improvements.

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