TWO SPECIAL CLASSES OF HAMILTONIAN GRAPH

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Abstract: We study two types of Hamiltonian graphs. We give a characterization to one type, and determine the minimum number of edges in the other type.
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1. Definitions

The graphs we consider are undirected, and without multiple edges or loops. Unless otherwise stated, we follow the notation of [1]. A Hamiltonian cycle (respectively, path) in a graph is a spanning cycle (path). A graph is Hamiltonian if it contains a Hamiltonian cycle. A Hamiltonian graph is said to be *weak* if adding any new edge will not creat a new Hamiltonian cycle. That is, if P is a Hamiltonian path in a weak Hamiltonian graph then the initial and final vertices of P must be adjacent. A Hamiltonian graph G is said to be *strong* if for every edge e in G there exists a hamiltonian cycle which does not use e. That is, G is hamiltonian, and $G - e$ is also hamiltonian for every e in G.

We shall characterize weak hamiltonian graphs and determine the minimum number of edges in a strong Hamiltonian graph.

Suppose G is a Hamiltonian graph on n vertices and let C be a Hamiltonian cycle. We define $i - chords$ of C as follows: an edge uv of G is an $i - chord$ of C if there exists a path on C, joining u and v, and of length i. Hence, an i-chord is also an $(n - i)$-chord. $N(v)$ denotes the neighbourhood of a vertex v.

2. Weak Hamiltonian Graphs

We characterize weak Hamiltonian graphs. An obvious example of weak Hamiltonian graphs is C_n, when the number of edges is minimum. K_n is also considered as a weak hamiltonian graph in the defined sense; where the number of edges is maximum obviously. We look for other important weak Hamiltonian graphs in the middle. In this section, we assume that G is weak Hamiltonian. We need the following lemmas.

Lemma 1. *If an i-chord is in E(G), then all of the i-chords are in E(G).*

Proof. Consider the Hamiltonian cycle in figure 1 where the vertices $v_0, v_1, v_2, ..., v_{n-1}$ are arranged in a clockwise order. Assume without loss of generality that $v_0v_i \in E(G)$. If $v_1v_{i+1} \notin G$ then there is a Hamiltonian path joining v_1 and v_{i+1}. Namely, $v_1v_2...v_iv_0v_{n-1}v_{n-2}...v_{i+1}$, which is a contadiction. Hence $v_1v_{i+1} \in G$. Proof follows now inductively. □

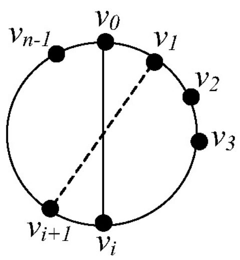

Figure 1

Lemma 2. *If there is an i-chord (i$\geq$ 2), then all of the 3-chords are present.*

Proof. We show that there is a 3-chord, and then we use lemma 1 to deduce that all of the 3-chords are present.

If there is an i-chord ($i \geq 2$), v_1v_{i+1} say.Then, by lemma 1 all of the i-chords are present. In particular v_2v_{i+2}. Therefore there is a Hamiltonian path joining v_0 and v_3, namely $v_0v_{n-1}v_{n-2}...v_{i+2}v_2v_1v_{i+1}v_iv_{i-1}...v_3$ (see figure 2). Hence the 3-chord v_0v_3 must be there. □

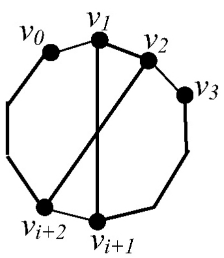

Figure 2

Lemma 3. *If there is an i-chord (i≥ 2), then all of the $(2k+1)$-chords are present where $k \geq 1$.*

Proof. By Induction on k. When $k = 1$, by lemma 2 all of the 3-chords are there. Suppose G contains all $(2l-1)$-chords, in particular v_0v_{2l-1}. Then we may use v_1v_{2l} and $v_{n-1}v_{2l-2}$ to deduce that there is a Hamiltonian path joining v_0 and v_{2l+1} (see figure 3). Hence G contains v_0v_{2l+1}, and by lemma 1 all of the $(2l+1)$-chords are present. Hence all of the odd chords are in $E(G)$. □

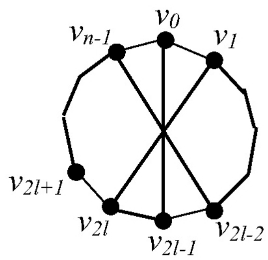

Figure 3

Proposition 4. *If G is a weak Hamiltonian graph on n vertices, where n is even, then G is isomorphic to C_n or contains $K_{\frac{n}{2},\frac{n}{2}}$ as a subgraph.*

Proof. If $G \ncong C_n$, then considering a Hamiltonian cycle C, there is an i-chord ($i \geq 2$). Hence by lemma 3 all odd chords are present. This proves that $K_{\frac{n}{2},\frac{n}{2}}$ is contained in G. □

We may use the following result which is due to Bondy, see [2].

Theorem 5. *If G is Hamiltonian and $|E(G)| \geq (\frac{n}{2})^2$, then either $G \cong K_{\frac{n}{2},\frac{n}{2}}$ or G is pancyclic.*

Corollary 6. *If G is a weak Hamiltonian graph on n vertices, where n is even, then G is isomorphic to either C_n, K_n, or $K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. Suppose $G \ncong C_n$. Then by proposition 4, G contains $K_{\frac{n}{2},\frac{n}{2}}$ as a subgraph, and so $|E(G)| \geq (\frac{n}{2})^2$. Now by theorem 5 we have either $G \cong K_{\frac{n}{2},\frac{n}{2}}$ or G is pancyclic. If $G \ncong K_{\frac{n}{2},\frac{n}{2}}$, then G is pancyclic. Therefore, if we consider a hamiltonian cycle C, then all odd chords are present (lemma 3), and all even chords are present as well (as G is pancyclic and using lemma 1). Hence $G \cong K_n$. □

Corollary 7. *If G is a weak Hamiltonian graph on n vertices, where n is odd, then G is isomorphic to either C_n or K_n.*

Proof. Suppose $G \ncong C_n$. then considering a Hamiltonian cycle C, there is an i-chord ($i \geq 2$). Hence by lemma 3 all odd chords are present. But since an i-chord can be regarded as an $(n-i)$-chord, then even chords can be regarded as odd chords as well. This means that all even chords must be present too, and $G \cong K_n$. □

Combining corollary 6 and corollary 7, we state our final result.

Proposition 8. *If G is a weak Hamiltonian graph on n vertices, then G is isomorphic to either C_n, K_n, or $K_{\frac{n}{2},\frac{n}{2}}$.*

3. Strong Hamiltonian Graphs

The obvious example here is K_n. Thus, strong Hamiltonian graphs of minimal size are to be sought. Suppose that $\varsigma(n)$ is the minimum number of edges in a strong Hamiltonian graph with n vertices. Clearly, the minimum degree δ is at least 3. Hence, if we could construct a class of cubic graphs or *nearly cubic* (A graph is nearly cubic if each vertex is of degree 3 except one vertex that is of degree 4), then we may conclude that $\varsigma(n)$ is $\frac{3n}{2}$ when n is even, and $\frac{3n+1}{2}$ when n is odd.

Proposition 9. *If G is a strong Hamiltonian graph on n vertices, then*
$$\varsigma(n) = \begin{cases} \frac{3n}{2} & \text{when } n \text{ is even} \\ \frac{3n+1}{2} & \text{when } n \text{ is odd} \end{cases}.$$

Proof. As we have mentiond, it is clear that the minimum degree δ is at least 3, and enough to find a cubic graph when n is even and nearly cubic when n is odd. Assume n is even, and consider the Mobius laddar in figure 4 where the vertices are arranged in a clockwise direction around the hamiltonian cycle C and $N(v_i) = \{v_{i-1}, v_{i+1}, v_{i+\frac{n}{2}}\}$. The indices are to be taken modulo $n-1$.

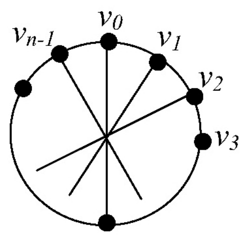

Figure 4

Using symmetry we may check upon deleting only two chords, a 1-chord (an edge of C) and an $\frac{n}{2}-$chord. The latter is trivial as C does use only 1-chords. So assume e is a 1-chord, v_0v_1 without loss of generality. Then, $v_0v_{n-1}v_{n-2}...v_{\frac{n}{2}+1}v_1v_2v_3...v_{\frac{n}{2}}v_0$ is a Hamiltonian cycle that does not use e.

Suppose now that n is odd. Let G be the nearly cubic graph modified from Mobius laddar (see figure 5), where the obvious Hamiltonian cycle is shown, $N(v_0) = \{v_1, v_{n-1}, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}\}$, and $N(v_i) = \{v_{i-1}, v_{i+1}, v_{i+\frac{n+1}{2}}\}$ $(i \neq 0)$. Indices are to be taken modulo $n-1$.

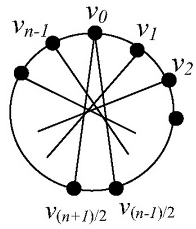

Figure 5

Once again, it is trivial to see that $G - e$ is still Hamiltonian when e is an $i-$chord ($i \neq 1$). Therefore, we may inspect only 1-chords. Note that there are $\frac{n-1}{2}$ of what we call the *cross configuration*, that is two crossed $\frac{n-1}{2}-$chords, Namely $\{v_{i-1}v_{i-1+\frac{n+1}{2}}, v_i v_{i+\frac{n-1}{2}}\}$, where $i \neq \frac{n+1}{2}$ (see figure 6).

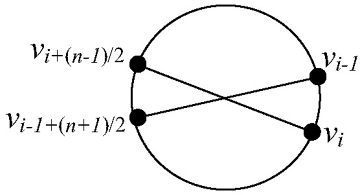

Figure 6

If $e = v_{i-1}v_i$ ($i \neq \frac{n+1}{2}$), then using a cross configuration, it is easy to spot a Hamiltonain cycle for $G - e$. Namely, $v_i v_{i+1} ... v_{i-1+\frac{n+1}{2}} v_{i-1} v_{i-2} ... v_{i+\frac{n+1}{2}} v_i$.

If $e = v_{\frac{n-1}{2}} v_{\frac{n+1}{2}}$, then taking all the i-chords ($i \neq 1$) together with these $\frac{n-1}{2}$ $1-$chords: $v_1v_2, v_3v_4, ..., v_{n-2}v_{n-1}$ gives us a Hamiltonian cycle for $G - e$. The case when $n = 9$ is shown in figure 7 where bold edges depict the Hamiltonian cycle.

This proves the result. □

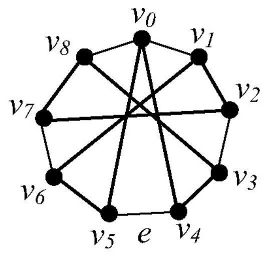

Figure 7

4. **Conclusion**

We have characterzied weak Hamiltonian graphs, and evaluated the minimum number of edges in a strong Hamiltonian graph.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Elsevier Science Publishing Co., Inc., 1976.

[2] B. Bollabas. *Extremal Graph Theory.* Academic Press Inc., London, 1978.