USEFUL MATHEMATICAL FORMULAS

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Series:

- $$\sum_{k=1}^{n} k = n(n+1)/2. \qquad \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6 \qquad \sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$$
- $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}, \; r \neq 1.$
- In general, $\sum_{k=p}^{q} r^k = \frac{r^{q+1}-r^p}{r-1}, \; r \neq 1.$
- $\sum_{k=p}^{\infty} r^k = \frac{r^p}{1-r}, \; r < 1.$
- $\sum_{k=1}^{n} k \cdot r^k = \frac{nr^{n+2}-(n+1)r^{n+1}+r}{(r-1)^2}, \; r \neq 1.$
- $\prod_{k=1}^{n} k^m = \left(\prod_{k=1}^{n} k\right)^m = (n!)^m.$

Logarithms:

- Let $a, b, c \in \mathbb{R}^+$ then $\log_b a = \frac{\log_c a}{\log_c b}.$
- $a = b^c \Leftrightarrow c = \log_b a.$
- $x^{\log_b y} = y^{\log_b x}.$
- $\log ab = \log a + \log b.$

Floor and Ceiling:

- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1.$
- $\lceil \log(n+1) \rceil = 1 + \lfloor \log n \rfloor.$

Limits:

- $\lim_{x \to a} f(x) = L \Rightarrow \forall \varepsilon \exists \delta \forall x (0 < |x - a| < \delta \to |f(x) - L| < \varepsilon).$
- L'Hospital's rule: if $\lim_{x \to a} f(x)/g(x) = \frac{0}{0}$ or $= \frac{\infty}{\infty}$ then $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x).$

Calculus:

Assume that u and v are functions of x.

$$\frac{d(cu)}{dx} = c\frac{du}{dx} \qquad \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx} \qquad \frac{d(u/v)}{dx} = \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right] \qquad \frac{d(e^{cu})}{dx} = ce^{cu}\frac{du}{dx}$$

$$\frac{d(c^u)}{dx} = (\ln c)c^u\frac{du}{dx} \qquad \frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$\int cu dx = c\int u dx \qquad \int (u+v)dx = \int u dx + \int v dx \qquad \int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad \int \frac{dx}{x} = \ln x$$

Combinatorial:

- $\binom{n}{k} = \binom{n}{n-k} = \frac{n}{k}\binom{n-1}{k-1} = \binom{n-1}{k} + \binom{n-1}{k-1} = \frac{n!}{(n-k)!k!}$.

Asymptotic Notation Definitions:

Consider functions $f, f_1, f_2,$ and g, g_1, g_2 their domain is the set of nonnegative integers; usually so is their range.

- $f = O(g) \Leftrightarrow \exists C, n_0 > 0 \ni f(n) \leq Cg(n), \forall n \geq n_0$.
- $f = \Omega(g) \Leftrightarrow \exists C, n_0 > 0 \ni f(n) \geq Cg(n), \forall n \geq n_0$.
- $$\begin{aligned} f = \Theta(g) \quad &\Leftrightarrow \quad \exists C_1, C_2, n_0 > 0 \ni C_1 g(n) \leq f(n) \leq C_2 g(n), \forall n \geq n_0 \\ &\Leftrightarrow \quad f = O(g) \text{ and } f = \Omega(g) \end{aligned}$$
- $f = o(g) \Leftrightarrow \lim_{n\to\infty} f(n)/g(n) = 0$.
- $f = \omega(g) \Leftrightarrow \lim_{n\to\infty} g(n)/f(n) = 0$.
- $f_1 = O(g_1)$ and $f_2 = O(g_2) \Rightarrow f_1 + f_2 = O(\max(g_1, g_2))$.
- $f_1 = O(g_1)$ and $f_2 = O(g_2) \Rightarrow f_1 f_2 = O(g_1 g_2)$.
- $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \Rightarrow \quad f = o(g), = O(g) \\ C > 0 & \Rightarrow \quad f = \Theta(g) \\ \infty & \Rightarrow \quad f = w(g), = \Omega(g). \end{cases}$$

Useful Theorem:

- Polynomial $\sum_{i=0}^{m} a_i n^i$ is $O(n^m)$, and if $a_m > 0 \Rightarrow \sum_{i=0}^{m} a_i n^i$ is $\Theta(n^m)$.

Recurrence Relation (Master Table):

Consider the recurrence,

$$T(n) = \begin{cases} T(1) & n = 1, \\ aT(n/b) + f(n) & n > 1. \end{cases}$$

Then $T(n) = n^{\log_b a}[T(1) + u(n)]$. For $u(n)$, compute $h(n) = f(n)/n^{\log_b a}$ then look at the corresponding entry in the table below,

$h(n)$	$u(n)$
$O(n^r)$, $r < 0$	$O(1)$
$\Theta(\log^k n)$, $k \geq 0$	$\Theta(\log^{k+1} n)$
$\Omega(n^r)$, $r > 0$	$\Theta(h(n))$