Exercise 1. (Fatou’s Lemma)

Let \((X, \mathcal{A}, \mu)\) be a measurable space and let \((f_n)_{n \in \mathbb{N}}\) and \(f\) \(\mu\)-measurable functions with the additional condition \(f\) and \(f_n\) belong to \(L^p(X)\) for all \(n \in \mathbb{N}\) and for all \(p \in [1, \infty)\). Assume that
\[
f = \lim_{n \to \infty} f_n \quad \mu - p.p. \quad \text{and} \quad \lim_{n \to \infty} \|f_n\|_p = \|f\|_p.
\]
We want to prove that \(f_n \to f\) in \(L^p(X)\) as \(n\) tends to \(\infty\).

1. Deduce from the convexity of the function \(\mathbb{R} \ni t \mapsto -|t|^p\) that
\[
\varphi_n := 2^{p-1}(|f_n|^p + |f|^p) - |f - f_n|^p \geq 0 \quad \mu - p.p.
\]
2. Apply Fatou’s Lemma to \(\varphi_n\) to prove that
\[
\limsup_n \int_X |f - f_n|^p d\mu = 0.
\]
3. Conclude.

Exercise 2. (Counter-example)

Let \((X, \mathcal{M}, \mu)\) be a measurable space and let \(f : X \to \mathbb{C}\) be a measurable function.

1. Show that if \(f \in L^1(\mu)\) then
\[
\lim_{n \to \infty} n\mu(\{|f| \geq n\}) = 0.
\]

2. Now we want to study whether the converse is true. Let \(g(x) = x \ln(x^{-1})\) defined on \([0, e^{-1}]\).

2. a. Show that
\[
\int_0^{e^{-1}} \frac{dx}{g(x)} = +\infty.
\]

2. b. Show that for \(n \in \mathbb{N}\); \(n\) large enough, there exists \(x_n \in [0, e^{-1}]\) such that \(\lim_{n \to \infty} x_n = 0\) and
\[
\{x \in [0, e^{-1}] : \frac{1}{g(x)} \geq n\} = [0, x_n].
\]

2. c. Deduce that
\[
n\mu(\{x \in [0, e^{-1}] : \frac{1}{g(x)} \geq n\}) = \frac{1}{|\ln(x_n)|}.
\]

2. d. Conclude.
Problem.

Part I: $L^\infty(X)$

Let $(X, \mathcal{M}, \mu)$ a measurable space. $\mu$ is being positive, and let $f : X \rightarrow \mathbb{C}$ be a measurable function such that $\mu(\{x \in X, f(x) \neq 0\}) > 0$. For all $p \in [1, +\infty)$, put

$$\varphi(p) = \int_X |f|^p d\mu \quad \text{and} \quad J = \{p \in [1, +\infty), \varphi(p) < +\infty\}.$$

1. Let $p_0 \leq p_1$ elements of $J$. Show that for all $\theta \in [0, 1]$, $p_\theta := (1-\theta)p_0 + \theta p_1$ belongs to $J$.

2. Show that $\varphi > 0$ on $J$ and that $\ln(\varphi)$ is convex on $J$.

3. Assume that there exists $r_0 \in [1, +\infty)$ such that $f \in L^{r_0}(\mu) \cap L^\infty(\mu)$. Prove that $f \in L^p(\mu)$ for all $p \in [r_0, +\infty)$, and that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

4. Suppose that there exists $r_0 \in [1, +\infty)$ such that $f \in L^p(\mu)$ for all $p \in [r_0, +\infty)$. Prove that if $f \not\in L^\infty(\mu)$,

$$\lim_{p \to \infty} \|f\|_p = +\infty.$$

Part II: Jensen inequality

Let $(X, \mathcal{M}, \mu)$ be a probability space, let $\varphi : \mathbb{R} \rightarrow \mathbb{R}^+$ twice differentiable with $\varphi'' \geq 0$, and let $f : (X, \mathcal{M}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable function such that

$$\int_X f(x) d\mu(x) < \infty.$$

1. Show that for all $z, y \in I$ (I interval) with $z \leq y$

$$\varphi(y) \geq \varphi(z) + \varphi'(z)(y-z).$$

2. Deduce that

$$\varphi(\int_X f(x) d\mu(x)) \leq \int_X \varphi \circ f(x) d\mu(x).$$

3. Prove that for all $f : [0, 1] \rightarrow \mathbb{R}$ such that $\int_0^1 f(x) d\mu(x) < \infty$

$$\left(\int_0^1 |f(x)| d\mu(x)\right)^2 \leq \int_0^1 f(x)^2 d\mu(x).$$

Part III: Probability space

Let $(X, \mathcal{M}, \mu)$ be a probability space and let $f \in L^\infty(\mu)$ non null function. Put $\alpha_n = \|f\|_{L^\infty(\mu)}^{-\frac{1}{n}}$. The purpose of this part is to prove that

$$(1) \quad \frac{\alpha_{n+1}}{\alpha_n} \rightarrow \|f\|_{L^\infty(\mu)} \quad \text{as} \quad n \quad \text{tends to} \quad +\infty.$$

1. Show that $0 < \alpha_n < \infty$ and that $\alpha_{n+1} \leq \|f\|_{L^\infty(\mu)} \alpha_n$.

2. Use Jensen inequality to prove that $\alpha_{n+1}^{\frac{1}{n}} \leq \alpha_{n+1} \leq \|f\|_{L^\infty(\mu)} \alpha_n$.

3. Deduce from Part I that

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} \rightarrow \|f\|_{L^\infty(\mu)}.$$

The end.