Answer of the first term exam M-204 (Borhen)

Question 1
$f(x,y)=\frac{\ln y}{x^2-1}$ is continuous in the region

$$R_1=\{(x,y)\in\mathbb{R}^2; x\neq\pm1, y>0\}.$$

$\frac{\partial f}{\partial y}(x,y)=\frac{1}{y(x^2-1)}$ is continuous in

$$R_2=\{(x,y)\in\mathbb{R}^2; x\neq\pm1, y\neq0\}.$$

Thus $f,\frac{\partial f}{\partial y}$ are continuous in R_1. Since $(-\frac{1}{2},2)\in D=\{(x,y)\in\mathbb{R}^2; |x|<1, y>0\}$ where f and $\frac{\partial f}{\partial y}$ are continuous, we conclude that the region which the initial value problem has a unique solution is

$$D=\{(x,y)\in\mathbb{R}^2; |x|<1, y>0\}.$$

Question 2
$\mu(x,y)=\frac{1}{x^2y}$ is an integrating factor, since the differential equation

$$\frac{1}{x^2y}\left[-y^2dx+(x^2+xy)dy\right]=0$$

is exact.
In fact: $-\frac{y}{x^2}dx+(\frac{1}{y}+\frac{1}{x})dy=0$, $M(x,y)=-\frac{y}{x^2}$, $N(x,y)=\frac{1}{y}+\frac{1}{x}$, $\frac{\partial M}{\partial y}(x,y)=-\frac{1}{x^2}$, $\frac{\partial N}{\partial x}(x,y)=-\frac{1}{x^2}$ then the DE is exact.
So there exists $F(x,y)$ that satisfies

$$\begin{cases}\frac{\partial F}{\partial x}(x,y) &= -\frac{y}{x^2} \quad (1)\\ \frac{\partial F}{\partial y}(x,y) &= \frac{1}{y}+\frac{1}{x} \quad (2)\end{cases}.$$

From first equation (1), there exists a function α such $F(x,y)=\frac{y}{x}+\alpha(y)$. From second equation (2), we have $\frac{\partial F}{\partial y}(x,y)=\frac{1}{x}+\alpha'(y)$. It follows from these equations that $\alpha'(y)=\frac{1}{y}$. So $\alpha(y)=\ln|y|+cst$ (cst is a real constant). Hence

$$F(x,y)=\frac{y}{x}+\ln|y|=cst$$

is the general solution of the differential equation.
Question 3

$$\begin{aligned}(x+2)^2y' &= 5-8y-4xy=5-4y(x+2)\\ y'+\frac{4y}{x+2} &= \frac{5}{(x+2)^2}. \quad (*)\end{aligned}$$

It is a Linear differential equation of first order. An integrating factor is

$$\mu(x)=e^{4\int\frac{dx}{x+2}}=e^{4\ln(x+2)}=(x+2)^4.$$

Multiply both sides of (*) by $\mu(x)$, we get :

$$\begin{aligned}\frac{d}{dx}\left[(x+2)^4y\right] &= 5(x+2)^2\\ (x+2)^4y &= \frac{5}{3}(x+2)^3+c, (c\in\mathbb{R})\\ y &= \frac{5}{3(x+2)}+c(x+2)^{-4}\end{aligned}$$

Question 4

$$\begin{aligned}(1-x^2)y'+xy-x\sqrt{y} &= 0, x\neq\pm1, y>0\\ y'+\frac{x}{1-x^2}y=\frac{x}{1-x^2}y^{1/2}, &\quad (I)\\ y^{-1/2}y'+\frac{x}{1-x^2}y^{1/2} &= \frac{x}{1-x^2}, \quad (II)\end{aligned}$$

Equation (I) is a Bernoulli's equation. Let $v = y^{1/2}$. Then $v' = \frac{1}{2}y^{-1/2}y'$. Equation (II) takes the form

$$v' + \frac{x}{2(1-x^2)}v = \frac{x}{2(1-x^2)}. \quad (III)$$

It is a linear differential equation. An integrating factor is $\mu(x) = e^{\frac{1}{2}\int \frac{xdx}{1-x^2}} = \frac{1}{(1-x^2)^{1/4}}$. Multiply both sides of (III) by $\mu(x)$, we get

$$\frac{d}{dx}\left(v.\frac{1}{(1-x^2)^{1/4}}\right) = \frac{x}{2(1-x^2)^{5/4}}.$$

By integration, we have: $v.\frac{1}{(1-x^2)^{1/4}} = \frac{1}{2}\int \frac{xdx}{(1-x^2)^{5/4}}$. Let $t = 1-x^2$. Then $dt = -2xdx$. So

$$\int \frac{xdx}{(1-x^2)^{5/4}} = -\frac{1}{2}\int \frac{dt}{t^{5/4}} = 2t^{-1/4} = \frac{2}{(1-x^2)^{1/4}} + cst.$$

Thus

$$\begin{aligned} \frac{v}{(1-x^2)^{1/4}} &= \frac{1}{(1-x^2)^{1/4}} + c, c \in \mathbb{R} \\ v &= 1 + c(1-x^2)^{1/4} \end{aligned}$$

Hence $\sqrt{y} = 1 + c(1-x^2)^{1/4}$. So $y = \left[1 + c(1-x^2)^{1/4}\right]$.

Question 5

Let $T(t)$ be the temperature of the hot iron rod at the time t and T_0 be the temperature of the air (temperature of surrounding). We have:

$$\begin{aligned} \frac{dT}{dt} &= k(T - T_0), \quad k \text{ is a constant of proportionality.} \\ \frac{dT}{T - T_0} &= kdt \\ \ln\frac{T - T_0}{c} &= kt, \quad c > 0 \\ T &= T_0 + ce^{kt} \end{aligned}$$

As $T_0 = 30°\text{C}$, $T(0) = 100°\text{C}$. Then $T(0) = 100 = 30 + c$. It follows that $c = 70$.
So $T(t) = 30 + 70e^{kt}$. As $T(15) = 70°\text{C}$, $70 = 30 + 70e^{15k}$. We get
$e^{15k} = \frac{4}{7} \Leftrightarrow k = \frac{\ln 4/7}{15} = -0.037$. Thus $T(t) = 30 + 70e^{-0.037t}$.

$$\begin{aligned} 40 &= 30 + 70e^{-0.037t} \\ e^{-0.037t} &= \frac{1}{7} \\ -0.037t &= \ln\frac{1}{7} = -1.946 \\ t &\approx 52.6 \quad \text{min} \end{aligned}$$

After 52.6 minutes the temperature of the hot iron rod will be 40°C.