Answer of the final exam M-203 (Dr Borhen)

Question 1

(a) Determine whether the following sequence $\{\sqrt{n^2+n}-n\}$ converges or diverges and if it converges find its limit.

$$\begin{aligned}\lim_{n\to\infty}\frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{(\sqrt{n^2+n}+n)} &= \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}-n}\\ &= \lim_{n\to\infty}\frac{n}{n(\sqrt{1+\frac{1}{n^2}}+1)}=\frac{1}{2}.\end{aligned}$$

(b) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$ converges or diverges.

As $\frac{\ln n}{n^3}<\frac{n}{n^3}=\frac{1}{n^2}$ which is a convergent series ($\sum\frac{1}{n^2}$ is convergent series by p-series test). Hence by basic comparison test $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$ is convergent.

(c) Find the interval of convergence and radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x-4)^n}{(n+1)^2}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{(n+1+1)^2}.\frac{(n+1)^2}{(-1)^n(x-4)^n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2(x-4)}{(n+2)^2}\right|\\ &= |x-4|\end{aligned}$$

For convergence $|x-4|<1\Leftrightarrow -1<x-4<1\Leftrightarrow 3<x<5$, $x\in(3,5)$.
At $x=3$, we have $\sum(-1)^n\frac{(-1)^n}{(n+1)^2}$ which is convergent.
At $x=5$, we have $\sum(-1)^n\frac{(1)^n}{(n+1)^2}$ which is convergent (absolutely convergent).
Hence the interval of convergent is $[3,5]$ and radius $R=\frac{5-3}{2}=1$.

Question 2

(a) Find Maclaurin's series for the function $f(x)=e^x$ and use this result to show that $\cosh(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$.

As $f(x)=e^x=f'(x)=f''(x)=...=f^{(n)}(x)=..$ then $f(0)=1=f'(0)=f''(0)=...=f^{(n)}(0)=....$ Hence we have

$$f(x)=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Now,

$$\begin{aligned}\cosh x &= \frac{e^x+e^{-x}}{2}=\frac{1}{2}[e^x+e^{-x}]\\ &= \frac{1}{2}\left([1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}+...]+[1-x+\frac{x^2}{2!}-...+(-1)^n\frac{x^n}{n!}+...]\right)\\ &= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+...\end{aligned}$$

(b) Evaluate the integral $\displaystyle\int_0^2\int_{y/2}^1\frac{\ln(1+x^2)}{1+x^2}dxdy$.

$$\begin{aligned}\int_0^2\int_{y/2}^1\frac{\ln(1+x^2)}{1+x^2}dxdy &= \int_0^1\int_0^{2x}\frac{\ln(1+x^2)}{1+x^2}dydx=\int_0^1\frac{\ln(1+x^2)}{1+x^2}\,[y]_0^{2x}\,dx\\ &= \int_0^1 2x\frac{\ln(1+x^2)}{1+x^2}dx=\frac{1}{2}\left[\ln(1+x^2)^2\right]_0^1=\frac{\ln 2}{2}.\end{aligned}$$

(c) Let R be the triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$. Find the surface area under the graph of $z=3x+y^2$ and over the region R.

We put $z = f(x,y) = 3x + y^2$. We have $f_x(x,y) = 3$ and $f_y(x,y) = 2y$. The surface area is

$$\begin{aligned} A &= \int\int_R \sqrt{1+(f_x)^2+(f_y)^2}dA \\ &= \int_0^1\int_0^y \sqrt{10+4y^2}dxdy = \frac{14^{3/2}-10^{3/2}}{12}. \end{aligned}$$

Question 3

`(a) Find the mass and centre of mass of the solid Q bounded by the graph of` $z = 4 - x^2 - y^2$ `and the` xy`-plane with a constant density` $\delta = 1$.

The mass m of the solid is:

$$\begin{aligned} m &= \int\int\int \delta dzdydx = \delta \int_0^{2\pi}\int_0^2\int_0^{4-r^2} rdzdrd\theta; \quad (r^2 = x^2+y^2) \\ &= \delta \int_0^{2\pi}\int_0^2 r\,[z]_0^{4-r^2} drd\theta = \delta\int_0^{2\pi}\int_0^2 r(4-r^2)drd\theta = \delta\int_0^{2\pi}\left[4\frac{r^2}{2}-\frac{r^4}{4}\right]_0^2 d\theta \\ &= \delta\int_0^{2\pi}(8-4)d\theta = \delta.4.2\pi = 8\pi\delta. \end{aligned}$$

By symmetry we have $\overline{x} = \overline{y} = 0$. We find $\overline{z}$ by the formula $\overline{z} = \frac{M_{xy}}{m}$ with

$$\begin{aligned} M_{xy} &= \int\int\int_Q \delta zrdzdydx = \delta\int_0^{2\pi}\int_0^2\int_0^{4-r^2} zrdzdrd\theta = \delta\int_0^{2\pi}\int_0^2 r\left[\frac{z^2}{2}\right]_0^{4-r^2} drd\theta \\ &= \delta\int_0^{2\pi}\int_0^2 \frac{1}{2}(16-8r^2+r^4)rdrd\theta = \delta\int_0^{2\pi}\frac{1}{2}\left[16\frac{r^2}{2}-8\frac{r^4}{4}+\frac{r^6}{6}\right]_0^2 d\theta = \frac{\delta}{2}\int_0^{2\pi}\frac{32}{3}d\theta = \delta\frac{32}{3}\pi. \end{aligned}$$

So $\overline{z} = \frac{\delta(32/3)\pi}{8\delta\pi} = \frac{4}{3}$. Hence the centre of mass of the solid Q has the coordinates $\left(0,0,\frac{4}{3}\right)$.

`(b) Evaluate the integral` $\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}dzdydx$ `by changing it to spherical coordinates.`

Using spherical coordinates, we get

$$\begin{aligned} \int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}dzdydx &= \int_0^\pi\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho\rho^2\sin\phi d\rho d\phi d\theta \\ &= \int_0^\pi\int_0^{\pi/4}\left[\frac{\rho^4}{4}\right]_0^{\sqrt{2}} \sin\phi d\phi d\theta \\ &= \int_0^\pi [-\cos\phi]_0^{\pi/4} d\theta = \int_0^\pi \frac{\sqrt{2}-1}{\sqrt{2}}d\theta = \frac{\pi}{2}(2-\sqrt{2}). \end{aligned}$$

`(c) Find the work done by the force` $\vec{F}(x,y,z) = -\frac{1}{2}x\vec{i} - \frac{1}{2}y\vec{j} + \frac{1}{4}\vec{k}$ `if it moves an object along the curve` C: $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 3\pi$.

The work done W of the force along the curve is

$$\begin{aligned} W &= \int_C M(x,y)dx + N(x,y)dy + P(x,y)dz = \int_C -\frac{1}{2}xdx - \frac{1}{2}ydy + \frac{1}{4}dz \\ &= \int_0^{3\pi} -\frac{1}{2}\cos t(-\sin t)dt - \frac{1}{2}\sin t(\cos t)dt + \frac{1}{4}dt = \int_0^{3\pi}\frac{1}{4}dt = \left[\frac{1}{4}\right]_0^{3\pi} = \frac{3\pi}{4}. \end{aligned}$$

Question 4

`(a) Use Green's theorem to evaluate the line integral` $\oint_C x^2y^2dx + (x^2-y^2)dy$ `where` C `is the boundary of the square with vertices` $(0,0), (1,0), (1,1)$, `and` $(0,1)$.

We have $\frac{\partial M}{\partial y}(x,y) = 2yx^2$ and $\frac{\partial N}{\partial x}(x,y) = 2x$. By Green's theorem

$$\begin{aligned}
\oint_C x^2y^2dx + (x^2-y^2)dy &= \int\int_R \left(\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y)\right) dA \\
&= \int_0^1\int_0^1 \left(2x - 2x^2y\right) dxdy \\
&= \int_0^1 \left[x^2 - \frac{2}{3}x^3y\right]_0^1 dy = \int_0^1 (1-\frac{2}{3}y)dy = \left[y - \frac{y^2}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.
\end{aligned}$$

(b) Use Divergence theorem to calculate the flux $\int\int_S \vec{F}.\vec{n}dS$ of the vector field $\vec{F}(x,y,z) = 2x^3\vec{i} + 2y^3\vec{j} + 2z^3\vec{k}$ through the sphere $x^2+y^2+z^2=4$.
Here we put $M(x,y) = 2x^3$, $N(x,y) = 2y^3$ and $P(x,y) = 2z^3$. We have:

$$\begin{aligned}
\int\int_S \vec{F}.\vec{n}dS &= \int\int\int_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV \\
&= \int\int\int_Q (6x^2+6y^2+6z^2)dV = 6\int\int\int_Q (x^2+y^2+z^2)dV \\
&= 6\int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^2\rho^2 \sin\phi d\rho d\phi d\theta = 6\int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^5}{5}\right]_0^2 \sin\phi d\phi d\theta \\
&= \frac{6}{5}.32\int_0^{2\pi} [-\cos\phi]_0^{\pi} d\theta = -\frac{192}{5}\int_0^{2\pi}(\cos\pi - \cos\theta)d\theta \\
&= (-\frac{192}{5})(-2)\int_0^{2\pi} d\theta = \frac{384}{5}(2\pi) = \frac{768}{5}\pi.
\end{aligned}$$

(c) Use Stoke's theorem to evaluate $\oint_C \vec{F}.d\vec{r}$, where $\vec{F}(x,y,z) = -y^2\vec{i} + z\vec{j} + x\vec{k}$, C is the boundary of the surface S bounded by the plane $2x+2y+z=6$ and the coordinate planes.
By Stoke's Theorem, we have:

$$\oint_C \vec{F}.d\vec{r} = \int\int_Q (curl\vec{F}).\vec{n}dS$$

$$\begin{aligned}
curl\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} \\
&= \vec{i}(-1) - \vec{j}(1) + \vec{k}(2y) = -\vec{i} - \vec{j} + 2y\vec{k} = M(x,y)\vec{i} + N(x,y)\vec{j} + P(x,y)\vec{k}.
\end{aligned}$$

So $M(x,y) = -1$,$N(x,y) = -1$ and $P(x,y) = 2y$. As the surface S is bounded, we can represent S as graph of $z = g(x,y)$ with $g(x,y) = 6-2x-2y$ on $R = \{(x,y); 0 \le x \le 3-y, 0 \le y \le 3\}$. So we have $g_x(x,y) = \frac{\partial g}{\partial x}(x,y) = -2$ and also $g_y = \frac{\partial g}{\partial y}(x,y) = -2$. It follows

$$\begin{aligned}
\int\int_S (curl\vec{F}).\vec{n}dS &= \int\int_R (-M.g_x - N.g_y + P)dA \\
&= \int\int_R (-2-2+2y)dA = \int_0^3\int_0^{3-y}(2y-4)dxdy \\
&= \int_0^3 [2yx - 4x]_0^{3-y} dy = \int_0^3 [2y(3-y) - 4(3-y)]\, dy \\
&= \int_0^3 (6y - 2y^2 - 12 + 4y)dy = \int_0^3 (10y - 2y^2 - 12)dy \\
&= 2\left[5\frac{y^2}{2} - \frac{y^3}{3} - 6y\right]_0^3 = 2\left[\frac{45}{2} - 27\right] = -9.
\end{aligned}$$