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Department of Mathematics
M-107

Time: 3 Hours
Full marks: 50

Final Exam
(Summer Semester, 1428-1429)

Question 1[6]:
Use Cramer's Rule to solve the system of linear equations:

$$\begin{cases} x+2y+3z &= 1 \\ 2x+5y+3z &= -2 \\ x+8z &= 8 \end{cases}.$$

Question 2[6]:
(a) Find the area of the triangle ABC where $A(1,2,0)$, $B(3,5,4)$ and $C(3,2,3)$.
(b) Find the angle between vectors $a = 6i-4j+2k$ and $b = 12i+15j-6k$.
(c) Find the distance from the point $L(3,3,1)$ to the line joining the points $M(1,1,1)$ and $N(1,2,3)$.
Question 3[12]:
(a) Identify the surface S: $4x^2+36y^2-9z^2-36=0$. Find its traces on the coordinate planes and sketch the surface.
(b) Find the tangential and normal components of acceleration for the curve $r(t) = e^t\, i + \sin t\; j + \tan t\; k$ at time t. Also find the curvature κ.
(c) Find the rectangular coordinates of the point given spherical coordinates: $P(3, \frac{\pi}{2}, \pi)$ and $Q(3, \pi, 0)$.
Question 4[8]:
(a) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2+y^2}{x^2-3y^2}$ does not exist.
(b) Use Chain Rule to show that:
i/ $f_{xy} = f_{yx}$ if $f(x,y) = \sin^2 x \cos y$.
ii/ $y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0$, if $w = f\,(x^2-y^2, y^2-x^2)$.
Question 5[4]: The electric potential V at (x,y,z) is given by

$$V(x,y,z) = x^4yz - xy^3 + z.$$

(a) Find the rate of change of V at $P(1,1,-3)$ in direction from P to origin.
(b) In what direction does V increases most rapidly ?

(c) What is the maximum rate of change at P ?

Question 6[6]

Find the points on the hyperboloid of two sheets $x^2 - 2y^2 - 4z^2 = 16$ at which the tangent plane is parallel to the plane $4x - 2y + 4z = 5$.

Question 7[8]

(a) Find all the critical points and indicate whether each point gives a local maximum, local minimum or whether it is a saddle point for the function $f(x, y) = 2x^4 - x^2 + y^2$.

(b) Use Lagrange multipliers to find the largest product of real numbers x,y and z, if $x + y + z^2 = 20$.

Dr Borhen

Correction of Final Exam (107M) (1428 - 1429H)

Exercise 1 [6pts]

$$\begin{cases} x + 2y + 3z = 1 \\ 2x + 5y + 3z = -2 \\ x + 8z = 8 \end{cases} \Leftrightarrow \quad AX = B \quad \text{with}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 5 \\ 1 & 0 \end{matrix} = [40 + 6] - [15 + 0 + 32] = 46 - 47 = -1.$$

So A has an inverse and this system has a unique solution: (1,5)

$$x = \frac{\det A_1}{\det A} = -\begin{vmatrix} 1 & 2 & 3 \\ -2 & 5 & 3 \\ 8 & 0 & 8 \end{vmatrix} = 0 \qquad (1,5)$$

$$y = \frac{\det A_2}{\det A} = -\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 3 \\ 1 & 8 & 8 \end{vmatrix} = -1 \qquad (1,5)$$

$$z = \frac{\det A_3}{\det A} = -\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -2 \\ 1 & 0 & 8 \end{vmatrix} = 1 \qquad (1,5)$$

Exercise 2: [6pts]

(a) The area of the triangle ABC is $\mathcal{A} = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$.

$\vec{AB} = (2,3,4)$
$\vec{AC} = (2,0,3)$;

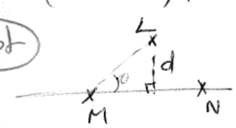
$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 0 & 3 \end{vmatrix} i - \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} j + \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} k = 9i + 2j - 6k.$$

So $\mathcal{A} = \frac{1}{2}\sqrt{9^2 + 2^2 + (-6)^2} = \frac{1}{2}\sqrt{121} = \frac{11}{2}\ cm^2$. (2pt)

(b) Let θ be the angle between a and b. We have: $a.b = |a||b|\cos\theta$.

So $\cos\theta = \dfrac{a.b}{|a|\,|b|}$. As $a.b = 0$ then $\theta = \frac{\pi}{2}$ $(a \perp b)$. (2pt)

(c) Let d be the distance. $d = \dfrac{\|\vec{ML} \times \vec{MN}\|}{\|\vec{MN}\|}$



$\vec{ML} = (2,2,0)$, $\vec{MN} = (0,1,2)$

$$\vec{ML} \times \vec{MN} = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 4i - 4j + 2k$$

Then $d = \dfrac{\sqrt{4^2 + (-4)^2 + 2^2}}{\sqrt{1^2 + 2^2}} = \dfrac{6}{\sqrt{5}} = \dfrac{6\sqrt{5}}{5}\ cm.$ (2pt)

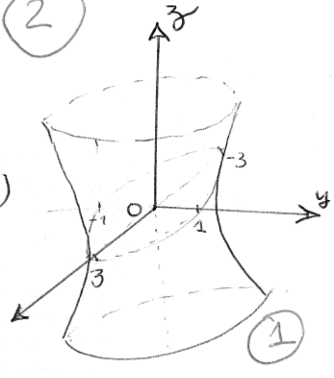
Exercise 3: [12pts]

(a) $S: 4x^2 + 36y^2 - 9z^2 = 36 \Leftrightarrow \dfrac{x^2}{3^2} + y^2 - \dfrac{z^2}{2^2} = 1.$

So S is a Hyperboloid of one sheet. (2pt)

traces - Coordinate plane	Description
xy-plane	$x^2/3^2 + y^2 = 1$ Ellipse
xz-plane	$x^2/3^2 - z^2/2^2 = 1$ Hyperbola
yz-plane	$y^2 - \frac{z^2}{2^2} = 1$ Hyperbola

2



1

(b) $r(t) = e^t\, i + \sin t\, j + \tan t\, k$

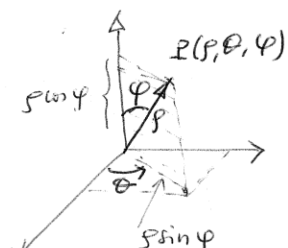
$r'(t) = e^t\, i + \cos t\, j + \sec^2 t\, k$ 1

$a(t) = r''(t) = e^t\, i - \sin t\, j + 2\sec^2 t \tan t\, k$ 1

We have: $= a_T\, T + a_N\, N$

- $a_T = \dfrac{r'(t) \cdot r''(t)}{\| r'(t) \|}$ 1 (tangential component of acceleration $a(t)$)
- $a_N = \sqrt{\|a\|^2 - a_T^2}$ (Normal component of acceleration) 1
- $\kappa = a_N \cdot \dfrac{1}{\| r'(t) \|^2}$ 1 (Curvature)

(c) From spherical to cartesian Coordinates



$$\begin{cases} x = \rho \sin\varphi \cos\theta & 0 \le \varphi \le \pi \\ y = \rho \sin\varphi \sin\theta \ ; & 0 \le \theta < 2\pi \\ z = \rho \cos\varphi \end{cases}$$

$P(3, \frac{\pi}{2}, \pi)$ has rectangular coordinate $(0,0,-3)$.

$Q(3, \pi, 0)$ has rectangular coordinate $(0,0,3)$. 2pt

Exercise 4 [8 pts]

(a) As $\lim\limits_{(x,0)\to(0,0)} \dfrac{3x^2+0}{x^2-0} = 3$ and $\lim\limits_{(0,y)\to(0,0)} \dfrac{0^2+y^2}{0-3y^2} = -\dfrac{1}{3}$ are different then $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2+y^2}{x^2-3y^2}$ does not exist. 2

(b) i/ $f(x,y) = \sin^2 x \cos y$ is a trigonometric function on 2 variable.

$f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}(x,y) = \dfrac{\partial}{\partial y}(2\sin x \cos x \cos y) = -2\sin x \cos x \sin y$

$f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial}{\partial x}(-\sin^2 x \sin y) = -2\sin x \cos x \sin y$

So $f_{xy} = f_{yx}$. 2

ii/ $w = f(x^2-y^2, y^2-x^2)$. Let $u(x,y) = x^2-y^2$ and $v(x,y) = y^2-x^2$. $w = f(u,v)$.

$$w < \begin{matrix} u < \begin{matrix} x \\ y \end{matrix} \\ v < \begin{matrix} x \\ y \end{matrix} \end{matrix}$$

2
$$\begin{cases} \dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial v} \cdot \dfrac{\partial v}{\partial x} = 2x f_u - 2x f_v. \\ \dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial w}{\partial v} \dfrac{\partial v}{\partial y} = -2y f_u + 2y f_v. \end{cases}$$

This imply: $y\dfrac{\partial w}{\partial x} + x\dfrac{\partial w}{\partial y} = 2xy f_u - 2xy f_v - 2xy f_u + 2xy f_v$
$= 0$. 2

Exercise 5: [4pts]

(a) The rate of change of V at P in direction from P to origin is $D_u V(1,1,-3)$ with $u = \dfrac{\overrightarrow{PO}}{\|\overrightarrow{PO}\|} = \left(\dfrac{-1}{\sqrt{11}}, -\dfrac{1}{\sqrt{11}}, \dfrac{3}{\sqrt{11}}\right)$.

$\overrightarrow{PO} = (-1,-1,3)$; $\|\overrightarrow{PO}\| = \sqrt{1+1+9} = \sqrt{11}$.

$$D_u V(1,1,-3) = \nabla V(1,1,-3) \cdot u$$
$$\nabla V(x,y,z) = (4x^3yz - y^3, x^4z - 3xy^2, x^4y+1)$$
$$\nabla V(1,1,-3) = (-13, -6, 2)$$

So $D_u V(1,1,-3) = \dfrac{13+6+6}{\sqrt{11}} = \dfrac{25}{\sqrt{11}}$. 2pt

(b) V increases rapidly in the direction of $\nabla V(1,1,-3) = -13i - 6j + 2k$ 1

(c) The maximum rate of change at P is $\|\nabla V(1,1,-3)\| = \sqrt{13^2+6^2+2^2} = \sqrt{209}$. 1

Exercise 6: [6pts]

1 * The normal vector of the plane $4x - 2y + 4z = 5$ is $n_1 = (4,-2,4)$.

1 * The normal vector of the tangent plane of S at $P(x,y,z)$ is

$$\nabla f(x,y,z) = (2x, -4y, -8z) \quad \text{where } f(x,y,z) = x^2 - 2y^2 - 4z^2 - 1$$

By hypothesis, there exits $k \in \mathbb{R}^*$ such that 1 $n_1 = k\nabla f(x,y,z)$

$$\begin{cases} 2kx = 4 \\ -4ky = -2 \\ -8kz = 4 \end{cases} \Rightarrow \begin{cases} x = 2/k \\ y = 1/2k \\ z = -1/2k \end{cases}$$

1 . As $x^2 - 4y^2 - 4z^2 = 16$ then $k = \pm \dfrac{1}{2\sqrt{2}}$. 2 So $\{(4\sqrt{2}, \sqrt{2}, -\sqrt{2}) ; (-4\sqrt{2}, \sqrt{2}, \sqrt{2})\}$

Exercise 7: [8 pts]

(a) As f is a polynomial function of 2 variables, the critical set is $C = \{(x,y) \ / \ \nabla f(x,y) = 0\}$

$$= \left\{(x,y) \ / \ \begin{array}{l} f_x(x,y) = 8x^3 - 2x = 0 \\ f_y(x,y) = 2y = 0 \end{array}\right\} = \left\{(x,y) \ / \ \begin{array}{l} 2x(4x^2-1) = 0 \\ y = 0 \end{array}\right\}$$

$$= \left\{ (0,0) \ ; \ (-\tfrac{1}{2}, 0) \ ; \ (\tfrac{1}{2}, 0) \right\}. \quad 1$$

The discriminant of f is:

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2}(x,y) \cdot \frac{\partial^2 f}{\partial y^2}(x,y) - \left[\frac{\partial^2 f}{\partial x \partial y}(x,y)\right]^2$$

$$= 2(24x^2 - 2) - 0 = 4(12x^2 - 1). \quad 1$$

. As $D(0,0) = -4 < 0$ then $(0,0,f(0,0)) = (0,0,0)$ is a saddle point for f. 1

- As $D(-1/2, 0) = 8 > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1/2, 0) = 4 > 0$ then f has a local minimum at $(-1/2, 0)$.
- As $D(\frac{1}{2}, 0) = 8 > 0$ and $\frac{\partial^2 f}{\partial x^2}(\frac{1}{2}, 0) > 0$ the f has a local minimum at $(\frac{1}{2}, 0)$.

1

(Note $f(-\frac{1}{2}, 0) = f(\frac{1}{2}, 0)$)!

(b) Let x, y and z the largest product of real numbers that satisfy $x + y + z^2 = 20$.

Let $f(x,y,z) = x \cdot y \cdot z$ and $g(x,y,z) = x + y + z^2 - 20$

By Lagrange Multipliers, there exits $\lambda \in \mathbb{R}$ such that

2 $\quad \begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \iff \begin{cases} yz = \lambda & (1) \\ xz = \lambda & (2) \\ xy = 2\lambda z & (3) \\ x + y + z^2 - 20 = 0 & (4) \end{cases}$

(1) and (2) $x = y$. By (3) $x^2 = 2\lambda z = 2xz^2$, $x = 2z^2$

By (4) we have $x + x + x/2 - 20 = 0 \Rightarrow x = 8 \Rightarrow z^2 = 4$. So $\{(8,8,-2), \boxed{(8,8,2)}\}$. 2

rejected $\downarrow$ (8,8,−2)