CHAPTER 2

SISO Linear System Analysis

4. Analysis of first order systems

A First order system is that whose output is modeled by a first order differential equation.

(1) first order lag (self-regulating)

\[
\begin{align*}
\frac{a_0}{a_1} \frac{dy}{dt} + a_1 y &= b u(t) \\
\frac{a_0}{a_1} \frac{dy}{dt} &= \frac{b}{a_1} u(t) \\
\tau \frac{dy}{dt} &= k_p u(t)
\end{align*}
\]

where \( \tau = \frac{a_0}{a_1} \); \( k_p = \frac{b}{a_1} \)

Taking Laplace transform with all variables are in deviation form we get:

\[
y(s) = \frac{k_p}{\tau s + 1} u(s)
\]

The term, \( \tau \) (time constant) and \( k_p \) (static gain) characterize the first order system.
**Time constant**: characterize the speed of response of a first-order system. It is a measure of the time necessary for a process to adjust to a change in the input.

**Static gain**: characterize the sensitivity of the output to the input signal.

**Dynamic response**:
The dynamic response for a step change in $u$ of magnitude of $a$: $u(s) = a/s$

$$y(t) = k_p a (1 - e^{-t/\tau})$$

The transient response for step change is shown in Figure (2).

![Figure 1: Response of a first-order lag system to step change in the input](image)

**Dynamic characteristics**:
- For a step change in the input, the process reaches a new steady state.
- The ultimate value of the output is $k_p a$, which can be found from the last Equation by setting $t \rightarrow \infty$
- The time constant, $\tau$ can be found by setting $t = \tau$ in the last equation which gives $y = 0.632 k_p a$. Then from Figure2, the time needed for $y$ to reach 0.632 $k_p a$ is $\tau$.
- The smaller the value of $\tau$, the steeper is the initial response of the output.
- The larger static gain of a process, the larger steady state value of its output for the same input change.
Figure 2: Effect of static gain, time constant on the response of first-order lag system

(2) Pure capacitive system (integrator)

\[ a_0 \frac{dy}{dt} = bu(t) \]

Example: Liquid storage tank with **fixed outlet flow**

\[ Ah(t) = F_i(t) - F \]

\[ h(s) = \frac{1}{As} F_i(s) - \frac{1}{As} F; \quad \Rightarrow \quad k_p = 1/A \]

\[ y(s) = \frac{k_p}{s} u(s) \]

**Dynamic response:**
For a step change in \( u \) of magnitude of \( a \): \( u(s) = a/s \)

\[ y(s) = \frac{k_p a}{s^2} \]

which have the following time response:

\[ y(t) = k_p at \]

The transient response looks like this:
• Apparently, the ultimate value for the output is not achievable, \( y \rightarrow \infty \) as \( t \rightarrow \infty \).

• A pure capacitive process causes serious control problem because it cannot balance itself.

• For small change in the input, the output grows continuously. This attribute knows as non-self-regulating (integrator) process.

5. Analysis of second order system

A second order system is that whose output is modeled by a second order differential equation.

\[
a_0 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = bu(t)
\]

\[
a_0 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + y = \frac{b}{a_2}u(t)
\]

\[
\tau^2 \frac{d^2 y}{dt^2} + 2\tau\zeta \frac{dy}{dt} + y = k_p u(t)
\]

\[
y(s) = \frac{k_p}{\tau^2 s^2 + 2\tau\zeta s + 1} u(s)
\]

\( \tau \) = natural period of oscillation (characteristic time constant)
\( \zeta = \text{damping ratio} \)

\( k_p = \text{static gain} \)

Sources of second order dynamics in the chemical industries come from a series of first-order systems, or a processing system with a controller.

**Dynamic response:**
For input step change of magnitude of \( a \), \( u(s) = a/s \) we have:

\[
y(s) = \frac{k_p a}{s(\tau^2 s^2 + 2\tau \zeta s + 1)} \equiv \frac{k_p a / \tau^2}{s(s - p_1)(s - p_2)}
\]

where

\[
p_1 = -\frac{\zeta}{\tau} + \frac{\sqrt{\zeta^2 - 1}}{\tau} \quad p_1 = -\frac{\zeta}{\tau} - \frac{\sqrt{\zeta^2 - 1}}{\tau}
\]

Case A: (over-damped) when \( \zeta > 1 \), we have two distinct and real poles

\[
y(t) = k_p a \left[ 1 - e^{-\zeta t / \tau} \right] \left( \cosh(\sqrt{\zeta^2 - 1} \frac{t}{\tau}) + \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh(\sqrt{\zeta^2 - 1} \frac{t}{\tau}) \right)
\]

Case B: (critically-damped) when \( \zeta = 1 \), we have two equal poles

\[
y(t) = k_p a \left[ 1 - \left(1 + \frac{t}{\tau}\right) e^{-t / \tau} \right]
\]

Case C: (under-damped) when \( \zeta < 1 \), we have two complex conjugate poles

\[
y(t) = k_p a \left[ 1 - \frac{1}{\sqrt{\zeta^2 - 1}} e^{-\zeta t / \tau} \sin(wt + \phi) \right]
\]

The dynamic response for all cases is shown in Figure (4)
The response of second order system to step change

- The over-damped response is sluggish and resembles a little the response of a first-order system. It becomes more sluggish with larger values of $\zeta$.
- The critically damped response is faster than the over-damped one.
- The under-damped response is faster than the others, but oscillates. The oscillatory behavior becomes pronounced with smaller values $\zeta$.

**Characteristics of under-damped system**

- Overshoot, $A/B$

\[ A / B = \exp \left( \frac{-\pi \zeta}{\sqrt{1 - \zeta^2}} \right) \]

- Decay ratio, $C/A$

\[ C / A = \exp \left( \frac{-2\pi \zeta}{\sqrt{1 - \zeta^2}} \right) \]

- Period of oscillation

\[ \omega = \text{radian frequency} = \frac{\sqrt{1 - \zeta^2}}{\tau} \]
$T = \text{period of oscillation} = \frac{2\pi \tau}{\sqrt{1 - \zeta^2}}$

- Rise time, $t_r$: is the time the output takes to first reach the new steady state value
- Process time, $t_p$: is the time required for the output to reach its first peak.
- Settling time, $t_s$: is the time required for the output to reach within $\pm 5\%$ of the new steady state value.

![Figure 5: characteristics of second-order under-damped response](image-url)
6. Delay time

The delay time known as transportation time is basically the time required for a material to move a specific distant. However, time delay is an inherent property of any chemical process.

\[ G(s) = e^{-\theta s} \]

Sources:
- The use of chromatography to measure composition of liquids or gases is another source of dead time.
- Flow in a long pipe.
- Time lag produced from staged processes.

Therefore, a first-order system with dead-time is:

\[ G(s) = \frac{Ke^{-\theta s}}{\tau_1 s + 1} \]

Therefore, a second-order system with dead-time is:

\[ G(s) = \frac{Ke^{-\theta s}}{\tau s^2 + 2\xi \tau s + 1} \]
7. Approximation of higher order systems

Systems with order higher than one can be represented by:

\[ G(s) = \prod_{i=1}^{n} G_i(s) = \frac{K}{\prod_{i=1}^{n}(\tau_i s + 1)} \]

It can be approximated by low order transfer function with dead time as follows:

\[ G(s) = \frac{Ke^{-\theta s}}{\tau_1 s + 1} \]

where \( \tau_1 \) is the dominant time constant and the dead time is:

\[ \theta = \sum_{i=2}^{n} \tau_i \]

8. Transfer function and Block diagram

Transfer function relates the process input to the process output in a linear mathematical function. Therefore,

**Transfer Function**: is the Laplace transform of the output, \( y(s) \) divided by the Laplace transform of the process input, \( u(s) \), with all initial conditions are zero (deviation form)

\[ G(s) = \frac{y(s)}{u(s)} \]

**Example**: For the storage tank we have:

\[ \frac{h(s)}{F(s)} = \frac{K}{\tau s + 1} \equiv G_p(s) ; \quad \text{with } \tau = AR, K = 1/R \]

**Example**: For the heated tank we have:
\[
\frac{T(s)}{Q(s)} = \frac{K}{\tau s + 1} \equiv G_p(s) \quad \text{with } \tau = V/F, \quad K = 1/F \rho C_p
\]

**Example:** For the CSTR we have:

\[
\frac{C_A(s)}{C_{A0}(s)} = \frac{K}{\tau s + 1} \equiv G_p(s) \quad \text{with } \tau = V/(F+Vk), \quad K = F/(F+Vk)
\]

### 8.1 Properties of Transfer Function

**Order:** The order of the system is the highest derivative of the output variable in the defining differential equation. For Transfer Function, it is the highest power of \( s \) in the denominator.

**Pole:** is the root of the denominator of the transfer function, i.e. the root of the characteristic polynomial. It directly determines:

- The stability of the system (positive poles)
- The potential of periodic transient (imaginary poles)

**Zero:** is the root of the numerator of the transfer function. It determines an inverse response (positive zero).

**Casuality:** A physical system is *causal* when the order of the denominator is greater than the numerator, and when the transfer function goes to 0 as \( s \to \infty \), the system is hence *strictly proper*. If the transfer function contains \( e^{\theta s} \) or the order of numerator is higher than the denominator, then the system is *non-casual* or *not realizable* because the current values of the system depends on the future values of the variables.

**Steady state gain:** is the steady state value of the transfer function, is evaluated by setting \( s = 0 \) in the stable transfer function.

### 8.2 Effect of poles and zeros

The poles and zeros of a transfer function affect the dynamic of a process.

Consider a particular transfer function:
\[ G(s) = \frac{K}{s(\tau_1 s + 1)(\tau^2 s^2 + 2\zeta \tau s + 1)} \]

The poles, i.e. the roots of the characteristic equation are:

\[ s_1 = 0 \]

\[ s_2 = -\frac{1}{\tau_1} \]

\[ s_3 = -\frac{\zeta}{\tau_2} + j\frac{\sqrt{1 - \zeta^2}}{\tau_2} \]

\[ s_4 = -\frac{\zeta}{\tau_2} - j\frac{\sqrt{1 - \zeta^2}}{\tau_2} \]

The poles can be represented in the complex plane as follows:

- Complex poles indicate the response will contain sine and cosine modes, i.e. will exhibit oscillation.
- Negative poles will result in a stable decaying response.
• Positive poles indicate that the response will have unstable mode.

The transfer function can be written as follows:

\[ G(s) = \frac{b_m (s - z_1)(s - z_2)\cdots(s - z_m)}{a_n (s - p_1)(s - p_2)\cdots(s - p_n)} \]

• Positive zero leads to inverse response.
• Zero-pole cancellation occurs when a zero has exactly the same numerical value as a pole.
• A zero can exert a profound effect on the coefficient of response mode.

8.3 Block diagrams
Block diagram is a graphical representation of transfer functions and their interactions.

Block diagram assist the engineer in determining the quantitative aspects of dynamic performance and in understanding the qualitative features of the system.

- \( y(s) = G(s)u(s) \)

- \( y(s) = A + B \)

- \( y(s) = G_1(s)G_2(s)u(s) \)
\[ y(s) = G_1(s)A - G_2(s)B \]

8.4 Input-output relation

Direct action: an increase in the input lead to an increase in the output.

Reverse action: an increase in the input lead to a decrease in the output.
A series of I/I blocks always result in I/I action:

\[ u \rightarrow \text{I/I} \rightarrow \text{I/I} \rightarrow y \]

An odd series of I/D blocks results in I/D action:

\[ u \rightarrow \text{I/I} \rightarrow \text{I/D} \rightarrow y \]

An even series of I/D blocks results in I/I action:

\[ u \rightarrow \text{I/D} \rightarrow \text{I/D} \rightarrow y \]
It is important that the overall system has an I/D action (negative feedback), therefore, the controller action must be adjusted such that the overall action of the closed-loop be I/D or reveres action.

Alternatively:

\[ k_c k_p = +ve \]