Chapter 10 LINEAR PROGRAMMING

10.1 INTRODUCTION

The subject of linear programming is to solve the following optimization problem. Find the vector $x = (x_1, x_2, \ldots, x_n)$ that maximizes (or minimizes) the linear objective function

$$z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \tag{10.1}$$

subject to the linear constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\le b_m \end{aligned} \tag{10.2}$$

In matricial form the problem can be written as:

$$\min_x (or \max_x) C^T x \tag{10.3}$$

subject to

$$Ax \le b \tag{10.4}$$

where

$$C = [c_1, c_2, \ldots, c_n]^T \tag{10.5}$$

is the vector of costs ,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{10.6}$$

is the matrix of constraints and

$$b = [b_1, b_2, \ldots, b_m]^T \quad (10.7)$$

is the right-hand side of the inequality constraints.

Linear programming (LP) problems are a special type of optimization problems studied in the previous chapter where the linear objective function (Eq 10.1) is convex and the linear constraints (Eq. 10.2) form a convex set. This implies that a local optimum will also be a global one. Linear programming arises often in chemical engineering optimization problems.The linear constraints can arise as a result of mass or allocation balances. They can also arise because of the limitations in raw materials or the limits imposed on a production line . Linear programming finds considerable applications in the oil and gas industry especially in the multi-plant production and distribution of refined oil products.

10.2 SIMPLEX METHOD

The most efficient iterative technique to solve the linear programming problem is the simplex algorithm. First the linear programming problem (Eqs. 10.1-10.2) is transformed into a standard form where there are only equality constrained. By introducing *slack* variables s_j, each inequality constraint

$$\sum_{i=1}^{n} a_{ij} x_j \leq b_i, \qquad j = 1, \cdots, m \quad (10.8)$$

can be transformed to an equality constraint

$$\sum_{i=1}^{n} a_{ij} x_j + s_j = b_i, \qquad j = 1, \cdots, m \quad (10.9)$$

with

$$s_j \geq 0, j = 1, \ldots, m \quad (10.10)$$

The detailed steps of the Simplex method are shown through the following example.

10.2.1 Problem Formulation

A chemical plant produces three different products P_1, P_2 and P_3 using two reactants R_1 and R_2. Table 10-1 shows the costs associated with each used raw material and its processing costs.

Table 10-1: Operational cost

Raw material	Cost (SAR/kg)	Processing costs (SAR/kg)
R_1	1.95	0.05
R_2	0.98	0.01

Table 10-2 shows, on the other hand, the sales price, the maximum allowable production and the composition of the reactants needed to obtain each product.

Table 10-2: Operational constraints

Product	Compositions per kg of product	Product Selling price SAR/kg	Maximum allowable production (kg/hr)
P_1	$0.78R_1$, $0.25R_2$	6	250
P_2	$0.15R_1$, $0.60R_2$	3	90
P_3	$0.07R_1$, $0.15R_2$	2	30

The problem we would like to solve is how to make use of the available raw materials in order to maximize the plant profit.

First let define the quantities of all the materials involved in the process

x_1 kg/hr of reactant R_1

x_2 kg/hr of reactant R_2

x_3 kg/hr of product P_1

x_4 kg/hr of product P_2

x_5 kg/hr of product P_3

- Objective function:

The profit function z is the difference between the income, cost of raw materials and the processing costs.

$$\text{Income: } I = 6x_3 + 3x_4 + 2\,x_5 \qquad (10.11)$$

$$\text{Cost of raw materials: } R_c = 1.95x_1 + 0.98x_2 \qquad (10.12)$$

$$\text{Operational cots: } O_c = 0.05x_1 + 0.01\,x_2 \qquad (10.13)$$

The profit function is, therefore

$$z = I - R_c - O_c \qquad (10.14)$$

or equivalently

$$z = (6x_3 + 3x_4 + 2x_5) - (1.95x_1 + 0.98x_2) - (0.05x_1 + 0.02\,x_2) \qquad (10.15)$$

that is

$$z = -2x_1 - x_2 + 6x_3 + 3x_4 + 2x_5 \qquad (10.16)$$

- Equality constraints

  The equality constraints arise from writing the material balances for each product. Using the data in Table 10-2 we have

$$\text{Product } P_1\text{: } 0.78x_1 + 0.25x_2 = x_3 \qquad (10.17)$$

$$\text{Product } P_2\text{: } 0.15x_1 + 0.6x_2 = x_4 \qquad (10.18)$$

$$\text{Product } P_3\text{: } 0.07x_1 + 0.15x_2 = x_5 \qquad (10.19)$$

- Inequality constraints

  The inequality constraints arise from the limits on the production levels for the three products. Using the data in Table [10.2], the following can be written:

$$\text{Product } P_1\text{: } x_3 \leq 250 \rightarrow 0.78x_1 + 0.25x_2 \leq 250 \qquad (10.20)$$

$$\text{Product } P_2\text{: } x_4 \leq 90 \rightarrow 0.15x_1 + 0.6x_2 \leq 90 \qquad (10.21)$$

$$\text{Product } P_3\text{: } x_5 \leq 30 \rightarrow 0.07x_1 + 0.15x_2 \leq 30 \qquad (10.22)$$

Since all the quantities involved in the problem are positive, non-negativity restrictions are imposed on all the variables i.e. $x_i \geq 0$ (i=1,5). Using the values of x_3, x_4 and x_5 Eqs. 10.17 up to 10.19, it is possible to write the objective function z in terms of the variables x_1 and x_2 only.

$$z = -2x_1 - x_2 + 6(0.78x_1 + 0.25x_2) + 3(0.15x_1 + 0.6x_2) + 2(0.07x_1 + 0.15x_2) \qquad (10.23)$$

$$z = 3.27x_1 + 2.6x_2 \qquad (10.24)$$

The linear programming problem can be written, therefore, as follows:

$$\max z = 3.27x_1 + 2.6x_2 \qquad (10.25)$$

Subject to

$$0.78x_1 + 0.25x_2 \leq 250 \qquad (10.26)$$

$$0.15x_1 + 0.6x_2 \leq 90 \qquad (10.27)$$

$$0.07x_1 + 0.15x_2 \leq 30 \qquad (10.28)$$

$$x_1 \geq 0 \qquad (10.29)$$

$$x_2 \geq 0 \qquad (10.30)$$

10.2.2 Graphical Representation

In a Cartesian system of co-ordinates x_1 and x_2 the straight lines forming the equality constraints derived from Eqs 10.26 to 10.30 are plotted in Figure 10-1.

Line (A) represents the equation

$$0.78x_1 + 0.25x_2 = 250 \qquad (10.31)$$

Line (B) represents the equation

$$0.15x_1 + 0.6x_2 = 90 \qquad (10.32)$$

while line (C) represents the equation

$$0.07x_1 + 0.15x_2 = 30 \tag{10.33}$$

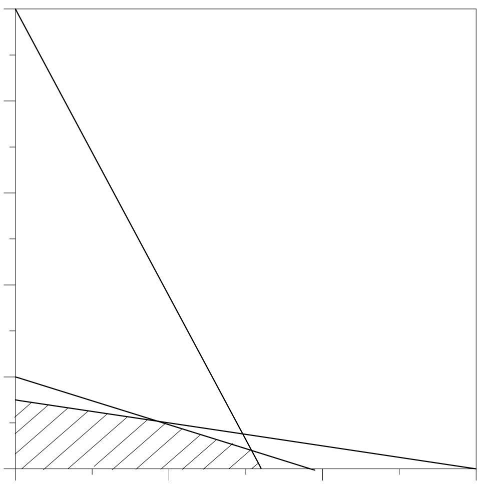

Figure 10-1:Plot of linear constraints

Each of these straight lines divides the plane in two half-planes. Only one half-plane satisfies the inequality constraint. Take for instance the first constraint, line (A) of (Eq. 10.31). The origin point $(x_1,x_2) = (0,0)$ which is located in the left half plane satisfies the first inequality constraint since by substituting in Eq 10.26 we have the relation

$$0.78*0 + 0.25*0 \leq 250 \tag{10.34}$$

which is true. Therefore, the left half plane is the admissible plane. Similarly we deal with the other equations. The region which remains admissible is bordered by sections of three straight lines. All the points within the region (shaded) including the borders satisfy the constraints and are called *feasible points*. The shaded region is called the *feasible region*. The next step is to find among all these feasible points, the point (or points) whose coordinates (x_1,x_2) maximizes the objective function $z = 3.27x_1 + 2.6x_2$.

In the next step we draw on the graph of feasible region, the lines on which the cost function is constant as shown by Figure 10-2. These lines are parallel to the line $3.27x_1 + 2.6x_2 = 0$. It can be seen from the graph that in order to maximize the objective function, we must go as far as we can, upward, without leaving the admissible region.

It is easy to visualize from the contour, which movements of the objective function along the constraints improve its value. From Figure 10-2, the optimum occurs at a corner of the feasible region (point OP) at values of x_1 = 301.5075 and x_2 = 59.2965.The optimum value of the objective function is, therefore z = 3.27*301.5075 + 2.6*59.2965 = 1140.1005 SAR/hr.

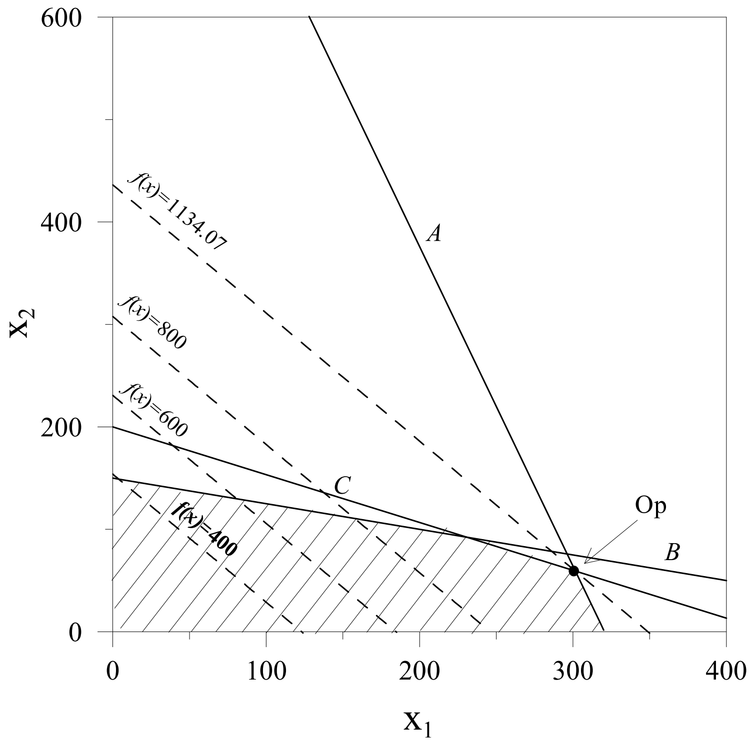

Figure 10-2: Plot of the cost function along with the linear constraints

Such a useful graphical visualization is possible only if the system of constraints consists of inequalities in two variables. But the results obtained for this example can be extended to the general case. In fact the fundamental result of linear programming optimization states that regardless of the feasible region, the maximum or minimum of a well-posed problem occurs always at a corner of the feasible region.

When faced with a linear system of order higher than two, a method more versatile than the graphical analysis should be therefore used to obtain the optimum. An efficient search technique must be implemented that progresses from one corner (vortex) to another. In the following section, the Simplex algorithm which is the most efficient known iterative techniques to solve the linear programming technique is presented.

10.3 SIMPLEX ALGORITHM

A practical implementation of the simplex algorithm is often accomplished through using a Tableau to represent the data.

First step:

The LP is transformed into the standard form (Eqs.10.9-10.10) by introducing the slack variables x_3, x_4 and x_5. The LP becomes then

$$\max z = 3.27x_1 + 2.6x_2 + 0x_3 + 0x_4 + 0x_5 \qquad (10.35)$$

subject to

$$0.78\,x_1 + 0.25x_2 + x_3 = 250 \qquad (10.36)$$

$$0.15x_1 + 0.60x_2 + x_4 = 90.0 \qquad (10.37)$$

$$0.07x_1 + 0.15x_2 + x_5 = 30.0 \qquad (10.38)$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \qquad (10.39)$$

In matricial form the LP is written as

$$\max z = [3.27\ 2.6\ 0\ 0\ 0]^{\mathrm{T}}\boldsymbol{x} \qquad (10.40)$$

subject to

$$\begin{bmatrix} 0.78 & 0.25 & 1 & 0 & 0 \\ 0.15 & 0.60 & 0 & 1 & 0 \\ 0.07 & 0.15 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 250 \\ 90 \\ 30 \end{bmatrix} \qquad (10.41)$$

$$\boldsymbol{x} = (x_1\ x_2\ x_3\ x_4\ \ x_5)^{\mathrm{T}} \geq 0 \qquad (10.42)$$

The LP has n = 5 variables (including the slack variables) and m = 3 equations, hence it has no unique solution. If the values of two ($n-m$) variables are fixed then we have a set of three variables and three equations and the system can be solved.

Step 2

We start by defining a *basic* solution that corresponds to a vortex of the feasible region. A *basic* solution is a solution for x obtained by solving for m variables in terms of the remaining $(n-m)$ variables and setting the $(n-m)$ variables equal to zero. The m nonzero variables are called the *basis* or *basic* variables. The $(n-m)$ variables are called the *nonbasic* variables.

For this specific example, the starting vector x_1 and x_2 (the nonbasic variables) are set to zero. The vector of basic variables (x_3, x_4, x_5) can be determined form the following equations:

$$0.78 \times 0 + 0.25 \times 0 + x_3 = 250 \tag{10.43}$$

$$0.15 \times 0 + 0.60 \times 0 + x_4 = 90.0 \tag{10.44}$$

$$0.07 \times 0 + 0.15 \times 0 + x_5 = 30.0 \tag{10.45}$$

This leads to

$$x_3 = 250, x_4 = 90, x_5 = 30 \tag{10.46}$$

and the objective function is:

$$z = 3.27*0 + 2.6*0 = 0 \tag{10.47}$$

This data is put in the first Tableau in the following form:

3.27	2.6	0	0	0	0
0.78	0.25	1	0	0	250
0.15	0.60	0	1	0	90
0.07	0.15	0	0	1	30
0	0	250	90	30	
x_1	x_2	x_3	x_4	x_5	

The Tableau is represented in this general form

C^T	Z
A	b
X	

The top raw of the tableau contains the coefficient c = (3.27,2.60,0,0,0) of the objective function. The current value of the objective function 0 is displayed in the top right corner. The next m rows in the Tableau represent the equality constraints (the matrix of Eqs. 10.36-10.39). The last row of the tableau contains the current value of vector x.

Step 3

Let us examine if the value of z can improve by allowing a nonbasic variable x_1 or x_2 to become a basic variable.We can see that a unit increase in x_1 is preferable to a unit increase in x_2 since the unit cost 3.27 (corresponding to x_1) is greater than 2.6 (corresponding to x_2). Hence, x_1 is candidate to become a basic variable. Next we determine which variable among (x_3, x_4, x_5) will leave the basis and be replaced by x_1 . Let held x_2 at zero and determine how much x_1 can be increased without violating the constraints.The equality constraints (Eqs 10.36 to 10.39) show that

$$250 - 0.78x_1 = x_3 \geq 0 \quad (10.48)$$

$$90 - 0.15x_1 = x_4 \geq 0 \quad (10.49)$$

$$30 - 0.07x_1 = x_5 \geq 0 \quad (10.50)$$

or equivalently

$$x_1 \leq 320.5128 \quad (10.51)$$

$$x_1 \leq 600.0 \quad (10.52)$$

$$x_1 \leq 428.5714 \quad (10.53)$$

Therefore, x_1 is allowed to increase only to the minimum of these values, i.e., 320.5128. This yields automatically x_3 = 0 and therefore x_3 is the variable that will leave the basis in favor of x_1. The rest of values x_4 and x_5 are obtained by Eqs. 10.48-10.54. The new vector is

$$x = (320.5128\ 0\ 0\ 41.9231\ 7.5641)^{T} \quad (10.54)$$

The new basic variables are therefore x_1, x_4, x_5.

Following this analysis a new Tableau is to be generated according to the following rules:

- Rule 1: The objective function is expressed only in terms of nonbasic variables
- Rule 2: Each basic variable occurs in only one row in the matrix of constraints

To satisfy rule 1 the basic variable x_1 is removed from the objective function. Equation (10.36) gives

$$x_1 = (250 - x_3 - 0.25x_2)/0.78 \qquad (10.55)$$

The objective function becomes

$$z = 3.27(250 - x_3 - 0.25x_2)/0.78 + 2.6x_2 = 1.5519x_2 - 4.1923x_3 + 1048.0769 \qquad (10.56)$$

To satisfy Rule 2, the new basic variable x_1 should appear only in one row. Taking (0.78) as the pivot element in the Tableau, the Gaussian elementary steps (Chapter 2) are performed as follows:

- Divide the first raw by 0.78 i.e. ($R_1/0.78 \rightarrow R_1$)
- Subtract 0.15 times the first row from the second row i.e. (R_2-0.15 $R_1 \rightarrow R_2$)
- Subtract 0.07 times the first row from the third row i.e. (R_3-0.07 $R_1 \rightarrow R_3$)

As a result, the second Tableau is obtained :

0	1.5519	−4.1923	0	0	1048.0768
1	0.3205	1.2821	0	0	320.5128
0	0.5519	−0.1923	1	0	41.9231
0	0.1276	−0.0897	0	1	7.5641
320.5128	0	0	41.9231	7.5641	
x_1	x_2	x_3	x_4	x_5	

The current *nonbasic* variables are x_2 and x_3. An increase in x_3 will decrease z because the cost associated with x_3 is negative, so x_2 is to become a basic variable. To find out which variable among ($x_{1,}$ $x_{4,}$ x_5) that will leave the basis we repeat the procedure of the previous step. Note that the procedure amounts to computing the ratios between the column of right hand side of the Tableau and the column of x_2 (the variable to join the basis). The minimum positive ratio corresponds to the variable that will leave the basis. The ratios are: 320.5128/0.3205, 41.9231/0.5519 and 7.5641/0.1275. The minimum ratio is the last one and it corresponds to the last constraint therefore it corresponds to x_5 which is the variable to leave the basis.

The value of x_2 is equal to the ratio 7.5641/0.1275=59.2965 and the new x-vector is

$$x = [301.5075\ 59.2965\ 0\ 9.1960\ 0]^T \tag{10.57}$$

The objective function in terms of the new nonbasic variables x_3 and x_5 is

$$z = 1.5519x_2 - 4.1923x_3 + 1048.0768 \tag{10.58}$$

The third constraint from the Tableau yields

$$x_2 = (7.5641 + 0.0897x_3 - x_5)/0.1276 \tag{10.59}$$

therefore

$$z = -3.1005x_3 - 12.1658x_5 + 1140.1005 \tag{10.60}$$

The third tableau is therefore

0	0	-3.1005	0	-12.1658	1140.1005
1	0	1.5075	0	−2.5126	301.5075
0	0	0.196	1	−4.3266	9.1960
0	1	0.7035-	0	7.8392	59.2965
301.5075	59.2965	0	9.1960	0	
x_1	x_2	x_3	x_4	x_5	

It is clear, since both the coefficients of z are negative, that the objective function can not be increased any further. The optimum values is, therefore, x_1 = 301.5075 and x_2 = 59.2965 and the maximum value of z is on the top right of the Tableau i.e. z=1140.1005.

The Tableau method as established is straightforward. The general procedure for forming the Tableau is summarized in the following steps:

- Select the nonbasic variables x_j whose coefficient in the objective function z is the largest positive .If all the coefficients in z are negatives then the current x is the optimum solution.
- Find the pivot element a_{kj} such that $b_i/a_{kj} = \min_i b_i/a_{ki}$. Set $x_j = b_k/a_{kj}$.
- Using the elementary Gauss operations and using the pivot element a_{kj} eliminate x_j from all the rows.
- These steps are repeated until an optimal solution is obtained or a conclusion is reached.

10.4 SPECIAL CASES

While the formation of the tableau is straightforward there are some cases where the formation of the Tableau can face problems.They are presented in the following section.

10.4.1 Multiple Solutions

This case is characterized by a zero at the bottom of a non-basic column .Consider the following example

$$\max z = 4x_1 + x_2 \tag{10.61}$$

subject to

$$4x_1 + x_2 \leq 56 \tag{10.62}$$

$$5x_1 + 3x_2 \leq 105 \tag{10.63}$$

$$x_1, x_2 \geq 0 \tag{10.64}$$

The first Tableau is

4	1	0	0	0
4	1	1	0	56
5	3	0	1	105
0	0	56	105	
x_1	x_2	x_3	x_4	

The pivot element is (4). Applying the rules of the Tableau yields:

0	0	1-	0	56
1	1/4	1/4	0	14
0	7/4	-5/4	1	35
14	0	0	35	
x_1	x_2	x_3	x_4	

This is a Tableau of an optimal solution, because there are no positive values in the first row. The optimal basic feasible solution is, therefore $x = [14\ 0\ 0\ 35]^T$ with $z = 56$.

However, there is a zero in the top of x_2 which is a non-basic variable and the objective function is $z = -x_3 + 56$ does not depend on x_2 (the other non-basic variable), therefore the variable x_2 can be changed to a basic variable and then from the next Tableau. Applying the Tableau rules to the pivot 7/4, the following Tableau is obtained:

0	0	−1	0	56
1	0	3/7	-1/7	9
0	1	-5/7	4/7	20
9	20	0	0	
X_1	x_2	x_3	x_4	

This is also an optimal Tableau with another optimal solution $x = [9\ \ 20\ 0\ 0]^T$. In fact there are infinite numbers of optimal solutions. This situation is displayed graphically in

Figure 10-3. It can be seen that the objective function lines and the first constraint *A* are parallel. This means that when sweeping across the feasible region with straight lines of equation $4x_1 + x_2 = z$, we reach eventually all the points of a whole stretch of the boundary and all the points of a whole stretch are optimal .

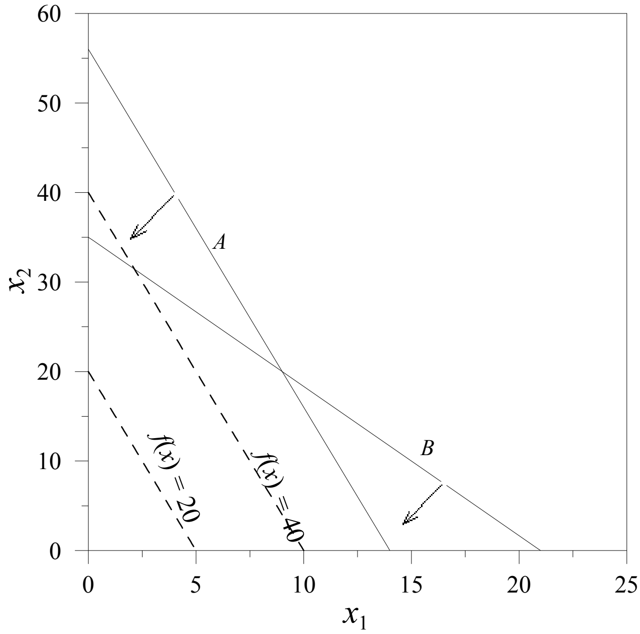

Figure 10-3: Example of multiple solutions

10.4.2 Unbounded Solution

Consider the case when trying to make a non-basic variable basic, but all coefficients of the basic variables are negative or zero and hence none is eligible as a pivot.

Consider the following example:

$$\max z = 5x_1 + x_2 \qquad (10.65)$$

Subject to

$$3x_1 - 2x_2 \le 6 \qquad (10.66)$$

$$-4x_1 + 2x_2 \le 4 \qquad (10.67)$$

$$x_1, x_2 \le 0 \qquad (10.68)$$

The first Tableau is

5	1	0	0	0
3	-2	1	0	6
-4	2	0	1	4
0	0	6	4	
x_1	x_2	x_3	x_4	

Using the pivot (3) we get the second Tableau

0	13/3	-5/3	0	10
1	-2/3	1/3	0	2
0	-2/3	4/3	1	12
2	0	0	16	
x_1	x_2	x_3	x_4	

It is required to make the variable x_2 basic since it has the highest cost coefficient, but all the elements are negative. The variable x_2 can be increased without bounds.The problem is, therefore, unbounded.The graph of Figure 10-4 shows that there is no bound in moving in a direction parallel to the cost function .

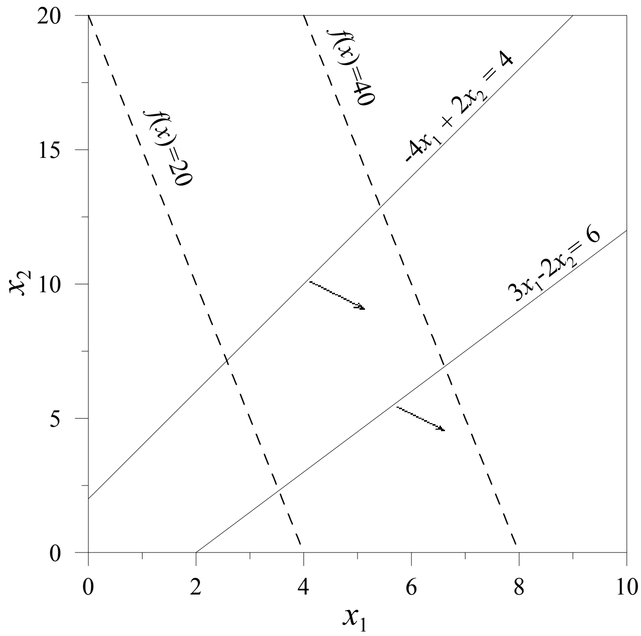

Figure 10-4: Unbounded Solution

10.4.3 No Obvious Feasible Solution Exists

In many instances, it is difficult to find a first feasible solution. Consider the following problem

$$\max z = 5x_1 + 3x_2 \tag{10.69}$$

Subject to

$$-4x_1 - 5x_2 \leq -10 \tag{10.70}$$

$$5x_1 + 2x_2 \leq 2 \tag{10.71}$$

$$3x_1 + 8x_2 \leq 12 \tag{10.72}$$

$$x_1, x_2 \geq 0 \tag{10.73}$$

Introducing the slack variables, the problem can be written as follows:

$$\max z = 5x_1 + 3x_2 \tag{10.74}$$

Subject to

$$-4x_1 - 5x_2 + x_3 = -10 \tag{10.75}$$

$$5x_1 + 2x_2 + x_4 = 2 \tag{10.76}$$

$$3x_1 + 8x_2 + x_5 = 12 \tag{10.77}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \tag{10.78}$$

It can be seen that the origin $(x_1,x_2) = (0,0)$ is not a feasible solution since, for the first constraint, this would require that $x_3 = -10$ which is a violation to the non-negativity of x_3. On the other hand, the variables $x_4 = 2$, $x_5 = 12$, for instance, are satisfactory. A common method to find an initial feasible solution is the *Big M* method .

Big M method

Big M method is used to overcome problems that arise due to none existence of feasible solution. To illustrate the method, reconsider the above example. Define a new artificial slack variable x_6 and introduce it in the first constraint equation in the following way

$$-4x_1 - 5x_2 + x_3 - x_6 = -10 \tag{10.79}$$

Note now that the origin $x_1 = 0$, $x_2 = 0$ is allowed since form Eq.(10.79) positive values of x_3 and x_6 can satisfy the equation $x_3 - x_6 = -10$. We can therefore start by solving the modified linear programming problem where the modified constraint of Eq.(10.79) replaces the original constraint of Eq.(10.75). However it must be made sure that in the final solution of this problem the value of x_6 is zero. This way the optimal values of the modified problem give a basic feasible solution to the original problem. In order to ensure that x_6 is zero, the objective function for the modified problem is changed to become:

$$z = 5x_1 + 3x_2 + Mx_6 \tag{10.80}$$

where M is a large positive number from which the method derived its name. If it is desirable to minimize an objective function, add M times the sum of the artificial numbers .

10.5 REVISED SIMPLEX METHOD

The simplex algorithm can be carried out more efficiently in a matrical form using the *revised simplex* method. At each iteration, the vector x is partitioned into a basis vector x_B (m, 1)and non-basis vector x_N(n-m,1).

$$x = [x_B \mid x_N] \tag{10.81}$$

The same is done for the matrix A

$$A = [A_B \mid A_N\,] \tag{10.82}$$

where A_B (m, m) is the basis matrix and A_N (n-m, m) is the nonbasis matrix. Similarly for the costs

$$C = [C_B \mid C_N] \tag{10.83}$$

The constraints equation,

$$Ax = b \tag{10.84}$$

is equivalent, then, to

$$A_B x_B + A_N x_N = b \tag{10.85}$$

The objective function can be written as:

$$z = c_B^{\mathrm{T}} x_B + c_N^{\mathrm{T}} x_N \tag{10.86}$$

From Eq.(10.85) we have

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N \tag{10.87}$$

Substituting Eq.(10.87) into Eq.(10.88) yields

$$z = c_B^{\mathrm{T}} A_B^{-1} b + (c^{\mathrm{T}}_N - c^{\mathrm{T}}_B A_B^{-1} A_N) x_N \tag{10.88}$$

The Tableau can be then put under the following suitable form

0	$c^T_N - c^T_B A_B^{-1} A_N$	$c_B^T A_B^{-1} b$
$I_{m\times m}$	$A_B^{-1} A_N$	$A_B^{-1} b$
$x_{B,1}, x_{B,1}, \ldots, x_{B,m}$	$x_{N,1}, x_{N,1}, \ldots, x_{N,n-m}$	

The iterations to update the Tableau are carried out in a similar way as presented in the direct Simplex method.

Example 10.1: Maximize the profit of a chemical plant

Reconsider the example for maximizing the profit of a chemical plant in section 10.2.1. In the first step,the basis variables are (x_3, x_4, x_5) and nonbasis (x_1, x_2). We have then

$$A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad and \quad A_N = \begin{bmatrix} 0.78 & 0.25 \\ 0.15 & 0.60 \\ 0.07 & 0.15 \end{bmatrix} \tag{10.89}$$

The costs are

$$C_B = [0\ 0\ 0]^T \quad \text{and } C_N = [3.27\ 2.6]^T \tag{10.90}$$

Since

$$A_B^{-1} = A_B \tag{10.91}$$

we have

$$A_B^{-1} A_N = A_N, \qquad A_B^{-1} b = b \quad and \quad C_N^T - C_B^T A_B^{-1} A_N = C_N^T \tag{10.92}$$

Looking for the largest positive element of C^T_N we realize that x_1 should enter the basis. We then perform the ratio test between $A_B^{-1} A_N$ and $A^{-1}{}_B b$ we see that the smallest positive ratio corresponds to the first constraint and therefore x_3 will leave the basis.

The new matrices A_B and A_N are:

$$A_B = \begin{bmatrix} 0.78 & 0 & 0 \\ 0.15 & 1 & 0 \\ 0.07 & 0 & 1 \end{bmatrix} \quad and \quad A_N = \begin{bmatrix} 0.25 & 1 \\ 0.60 & 0 \\ 0.15 & 0 \end{bmatrix} \tag{10.93}$$

With

$$C^T{}_B = [3.27 \quad 0 \quad 0] \quad and \quad C^T{}_N = [2.6 \quad 0] \tag{10.94}$$

The products $A_B^{-1}A_N$, $A_B^{-1}b$ and the relative costs $C_N^{\ T} - C_B^{\ T}\ A_B^{-1}A_N$ are:

$$A_B^{-1}A_N = \begin{bmatrix} 0.3205 & 1.2821 \\ 0.5519 & -0.1923 \\ 0.1276 & -0.0897 \end{bmatrix} and \ A_B^{-1}b = \begin{bmatrix} 320.5128 \\ 41.9231 \\ 7.5641 \end{bmatrix} \tag{10.95}$$

$$C^T{}_N - C^T{}_B A^{-1}{}_B A_N = [1.5519 \quad -4.1923] \tag{10.96}$$

From the components of the relative cost we see that the variable x_2 should enter the basis. We then perform the ratio test between $A_B^{-1}A_N$ and $A^{-1}{}_B b$ we see that the smallest positive ratio corresponds to the second constraint and therefore x_4 will leave the basis.

The new matrices A_B and A_N are:

$$A_B = \begin{bmatrix} 0.78 & 0.25 & 0 \\ 0.15 & 0.60 & 1 \\ 0.07 & 0.15 & 0 \end{bmatrix} \quad and \quad A_N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{10.97}$$

With

$$C_B^{\ T} = [3.27 \quad 2.6 \quad 0] \quad and \quad C_N^{\ T} = [0 \quad 0] \tag{10.98}$$

The products $A_B^{-1}A_N$, $A_B^{-1}b$ and the relative costs $C^T{}_N - C^T{}_B A^{-1}{}_B A_N$ are

$$A_B^{-1}A_N = \begin{bmatrix} 1.5075 & -2.5126 \\ -0.7035 & 7.8392 \\ 0.1960 & -4.3266 \end{bmatrix} \; and \; A_B^{-1}b = \begin{bmatrix} 301.5075 \\ 59.2965 \\ 9.1960 \end{bmatrix} \tag{10.99}$$

$$C^T{}_N - C^T{}_B A^{-1}{}_B A_N = [-3.1005 \quad -12.1658] \tag{10.100}$$

The two components of the relative cost vector are negative indicating that the optimum solution has been reached. The optimum objective function is $C^T{}_B A^{-1}{}_B A_N = 1140.1005$ Using the matrical form introduced, the final tableau can be then put suitably under the following form which is similarly to the final Tableau in the direct simplex method.

0	0	0	-3.1005	-12.1658	1140.1005
1	0	0	1.5075	−2.5126	301.5075
0	1	0	0.7035-	7.8392	59.2965
0	0	1	0.196	−4.3266	9.1960
301.5075	59.2965	9.1960	0	0	
x_1	x_2	x_4	x_3	x_5	

10.6 SENSITIVITY ANALYSIS

After a linear programming problem has been solved completely it is often useful to investigate the effects of changes of some problem parameters on the optimal solution.The following changes will be discussed in this section:

- Changes in the components of the right hand side original vector.
- Changes in the cost coefficients of the objective function.

This analysis is important because the costs or the limits in the constraints may be poorly known; and even if they are known accurately the effects of expanding capacity or the changes in the costs may be needed to be examined. Calculation of the change in the value of the optimal solution in response to changes in coefficients of the objective function or in the constraints is known as sensitivity analysis . The advantage of the simplex method is that sensitivity information can be obtained from the Tableau without re-computing the optimal solution.

10.6.1 Changes In The Coefficients Of The Right Hand Side Vector

Examination of the final Tableau in matricial form provides the following results on the sensitivity analysis.

- The columns of the basis A_B^{-1} give the rates of changes of the solution component of the basic vector x_B for an increase in the corresponding components of the right-hand side vector.
- The elements of the vector $C_B A_B^{-1}$ give the per unit rate of change in the objective function for a unit change in the right hand side of a constraint. These rates of changes are also called *shadow prices*.
- For a maximization problem the sign of the element of $C_B B^{-1}$ indicates the direction of changes (increase or decrease) of the particular right hand side. A zero value indicates no effect.

Example 10.2 : Repeat Example 10.1

Consider the final Tableau in the previous example. The matrix A_B^{-1} and vector $C_B A_B^{-1}$ are:

$$A^{-1}{}_B = \begin{bmatrix} 1.5075 & 0 & -2.5126 \\ -0.7035 & 0 & 7.8392 \\ 0.1960 & 1 & -4.3266 \end{bmatrix} \qquad (10.101)$$

$$C_B{}^T A^{-1}{}_B = \begin{bmatrix} 3.1005 & 0 & 12.1658 \end{bmatrix} \qquad (10.102)$$

The following can be observed. The first column of A_B^{-1} i.e. [1.5075 -0.7035 0.1960] provide the rate of changes $\Delta x_i/\Delta b_1$ of the solution x_1, x_2 and x_4 for a unit increase in the right-hand-side of the first constraint (b_1). Thus, if the number 250 in the first constraint were increased by 1 unit, x_1 is increased by 1.5075 from its value of 301.5075; x_2 is decreased by amount of -0.7035 from its value of 59.2965 and x_4 is increased by amount of 0.1960 from its value of 9.1960. The constraint 250 can not be however increased indefinitely. In fact the amount of increase is limited by 59.2965/0.7035 = 84.2878. Beyond this value the variable x_2 becomes negative. The decrease in b_1 is also limited by max (−301.5075/1.5075, −9.1960/0.1960) = −9.1960/0.1960 = −46.9184 to maintain the positivity of x_1 and x_4. Therefore, the first hand side element is bounded by

$$250 - 46.9184, \text{ and } 250 + 84.2878 \tag{10.103}$$

That is by

$$203.0816 \text{ and } 334.2878 \tag{10.104}$$

The same analysis can be carried out for the second column of A_B^{-1}. Note that the second column indicates that the variables x_1 and x_2 remain unaltered for any increase in the right hand side of the second constraint.

The second right hand side should be greater than −9.1960/1 = −9.1960 to maintain the positivity of x_4 .The second right hand side is limited between

$$|-9.1960, \infty| \tag{10.105}$$

The third column of A_B^{-1} shows that the third right hand side is to be bounded by − 59.2965/7.8392 and min (301.5075/2.5126, 9.1960/4.3266) = 2.1254. The third hand side is limited, then, by:

$$[30 - 7.5641 \quad \text{and} \quad 30 + 2.1254] \tag{10.106}$$

Or

$$[22.4359, \ 32.1254] \tag{10.107}$$

The term $C_B A_B^{-1}$ is [3.1005 0 12.1658]. The first component indicates that the profit will be increased at a marginal rate of 3.1005 per unit change in the right-hand-side term of 250. However there are limits of this increase as indicated previously . The second term indicates that lowering or relaxing the second constraint will not affect the optimal objective function as it was shown previously. The third component of indicates that the profits will be increased at a marginal rate of 12.1658 per unit change in the right-hand-side term of 30.

10.6.2 Effect Of Change In The Cost Function Coefficients

To analyze the effects of changes in the coefficients of the objective function, the relative costs, i.e., coefficient at the bottom of the Tableau, have to be recomputed. Consider the change in the cost of the first variable $c_1 = 3.27 + \theta$.The relative costs are $C^T{}_N - C^T{}_B A^{-1}{}_B A_N$

with

$$C_N{}^T = [0\ 0] \quad \text{remains unchanged} \tag{10.108}$$

and

$$C^T{}_B = [3.27+\theta\ 0\ 2.6] \tag{10.109}$$

The relative costs become:

$$[-3.1005-1.5075\theta \quad -12.1658+2.5126\theta]^T \tag{10.110}$$

and the value of the objective function becomes :

$$z = c_B B^{-1} b = 1140.1005+301.5075\,\theta \tag{10.111}$$

For the solution to remain optimal all the relative costs must remain negative. That is

$$-3.1005-1.5075\theta \le 0 \rightarrow \theta > -2.0567 \tag{10.112}$$

$$-12.1658+2.5126\theta \le 0 \rightarrow \theta < 4.8420 \qquad 13.113)$$

Therefore, for the range

$$-2.0567 \le \theta \le 4.8420 \tag{10.114}$$

or equivalently

$$3.27 - 2.0567 \le c_1 \le 3.27 + 4.8420 \tag{10.115}$$

$$1.2133 \le c_1 \le 8.1120 \tag{10.116}$$

the change in c_1 within this range will not affect the change in the basic variables in the optimal solution although the objective function will increase by amount of 301.50750θ (Eq. 11.111)

The same analysis for the change in the second coefficient $c_2 = 2.6 + \theta$ can be studied. The relative cost functions are

$$[-3.1005+0.70350\theta \quad -12.1658-7.83920\theta]^T \tag{10.117}$$

and the objective function becomes:

$$1140.1005+59.29650\theta \tag{10.118}$$

As long as the relative costs are negatives there will be no change in the basic variables of the optimal solution. This is equivalent to

$$-1.5520 \leq \theta \leq 4.4072 \tag{10.119}$$

which is equivalent to

$$2.6 - 1.5520 \leq c_2 \leq 2.6 + 4.4072 \tag{10.120}$$

Or

$$1.0480 \leq c_2 \leq 7.0072 \tag{10.121}$$

10.7 DUALITY

For any given Linear Programming problem that we denote *primal* we can define a *dual* problem as follows:

Primal problem :

$$\text{maximize } z = c_1x_1 + c_2\,x_2 + \ldots + c_nx_n \tag{10.122}$$

Subject to the following constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}\,x_n \leq b_1 \tag{10.123}$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}\,x_n \leq b_2 \tag{10.124}$$

$$\vdots \tag{10.125}$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}\, x_n \leq b_m \tag{10.126}$$

and to constraints that are:

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \tag{10.127}$$

The dual problem is defined as follows:

Dual problem:

$$\text{minimize } z = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m \tag{10.128}$$

subject to the following constraints

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}\, y_m \geq c_1 \tag{10.129}$$

$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}\, y_m \geq c_2 \tag{10.130}$$

$$\vdots \tag{10.131}$$

$$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}\, y_m \geq c_n \tag{10.132}$$

and to constraints that are:

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0 \tag{10.133}$$

Note that the following changes have been made:

- The objective function coefficients of the primal problem have become the right-hand side constants of the dual. Similarly the right hand side constants of the primal have become the cost coefficients of the duals.
- The maximization problem of the primal is the minimization problem for the dual.
- The matrix A^T is used to form the dual instead of the matrix A.
- The sense of inequalities is reversed

A fundamental result of linear programming states in particular that the maximum of the objective function of the maximizing problem is equal to the minimum of that of its dual provided a finite optimum exists to both problems. Moreover, the components of the vector y which are the solutions to the dual problem are the shadow prices obtained by solving the primal problem.

10.8 OTHER SOLUTION TECHNIQUES.

Some techniques solve the linear programming problem through the use of both the primal and dual formulation. The dual simplex method is one example of these solutions. See references [22,49] for more detail.

10.9 IMSL ROUTINES

Routine	Features
DLPRS	Solve a linear programming problem via the revised simplex algorithm.