ode45, ode23, ode113, ode15s, ode23s, ode23t, ode23tb

Solve initial value problems for ordinary differential equations (ODEs)

Syntax

```
[T,Y] = solver(odefun,tspan,y0)
[T,Y] = solver(odefun,tspan,y0,options)
[T,Y] = solver(odefun,tspan,y0,options,p1,p2...)
[T,Y,TE,YE,IE] = solver(odefun,tspan,y0,options)
sol = solver(odefun,[t0 tf],y0...)
```

where `solver` is one of `ode45`, `ode23`, `ode113`, `ode15s`, `ode23s`, `ode23t`, or `ode23tb`.

Arguments

`odefun`	A function that evaluates the right-hand side of the differential equations. All solvers solve systems of equations in the form $y' = f(t, y)$ or problems that involve a mass matrix, $M(t, y)y' = f(t, y)$. The `ode23s` solver can solve only equations with constant mass matrices. `ode15s` and `ode23t` can solve problems with a mass matrix that is singular, i.e., differential-algebraic equations (DAEs).
`tspan`	A vector specifying the interval of integration, `[t0,tf]`. To obtain solutions at specific times (all increasing or all decreasing), use `tspan = [t0,t1,...,tf]`.
`y0`	A vector of initial conditions.
`options`	Optional integration argument created using the `odeset` function. See `odeset` for details.
`p1,p2...`	Optional parameters that the solver passes to `odefun` and all the functions specified in `options`..

Description

`[T,Y] = solver(odefun,tspan,y0)` with `tspan = [t0 tf]` integrates the system of differential equations $y' = f(t, y)$ from time `t0` to `tf` with initial conditions `y0`. Function `f = odefun(t,y)`, for a scalar `t` and a column vector `y`, must return a column vector `f` corresponding to $f(t, y)$. Each row in the solution array `Y` corresponds to a time returned in column vector `T`. To obtain solutions at the specific times `t0`, `t1`, `...`, `tf` (all increasing or all decreasing), use `tspan = [t0,t1,...,tf]`.

`[T,Y] = solver(odefun,tspan,y0,options)` solves as above with default integration parameters replaced by property values specified in `options`, an argument created with the `odeset` function. Commonly used properties include a scalar relative error tolerance `RelTol` (`1e-3` by default) and a vector of absolute error tolerances `AbsTol` (all components are `1e-6` by default). See `odeset` for details.

`[T,Y] = solver(odefun,tspan,y0,options,p1,p2...)` solves as above, passing the additional parameters `p1,p2...` to the function `odefun`, whenever it is called. Use `options = []` as a place holder if no options are set.

`[T,Y,TE,YE,IE] = solver(odefun,tspan,y0,options)` solves as above while also finding where functions of (t,y), called event functions, are zero. For each event function, you specify whether the integration is to terminate at a zero and whether the direction of the zero crossing matters. Do this by setting the `'Events'`

property to a function, e.g., `events` or `@events`, and creating a function [`value`,`isterminal`,`direction`] = `events`(`t`,`y`). For the `i`th event function in `events`:

- `value(i)` is the value of the function.
- `isterminal(i)` = `1` if the integration is to terminate at a zero of this event function and `0` otherwise.
- `direction(i)` = `0` if all zeros are to be computed (the default), `+1` if only the zeros where the event function increases, and `-1` if only the zeros where the event function decreases.

Corresponding entries in `TE`, `YE`, and `IE` return, respectively, the time at which an event occurs, the solution at the time of the event, and the index `i` of the event function that vanishes.

`sol = solver(odefun,[t0 tf],y0...)` returns a structure that you can use with `deval` to evaluate the solution at any point on the interval `[t0,tf]`. You must pass `odefun` as a function handle. The structure `sol` always includes these fields:

`sol.x`	Steps chosen by the solver.
`sol.y`	Each column `sol.y(:,i)` contains the solution at `sol.x(i)`.
`sol.solver`	Solver name.

If you specify the `Events` option and events are detected, `sol` also includes these fields:

`sol.xe`	Points at which events, if any, occurred. `sol.xe(end)` contains the exact point of a terminal event, if any.
`sol.ye`	Solutions that correspond to events in `sol.xe`.
`sol.ie`	Indices into the vector returned by the function specified in the `Events` option. The values indicate which event the solver detected.

If you specify an output function as the value of the `OutputFcn` property, the solver calls it with the computed solution after each time step. Four output functions are provided: `odeplot`, `odephas2`, `odephas3`, `odeprint`. When you call the solver with no output arguments, it calls the default `odeplot` to plot the solution as it is computed. `odephas2` and `odephas3` produce two- and three-dimnesional phase plane plots, respectively. `odeprint` displays the solution components on the screen. By default, the ODE solver passes all components of the solution to the output function. You can pass only specific components by providing a vector of indices as the value of the `OutputSel` property. For example, if you call the solver with no output arguments and set the value of `OutputSel` to `[1,3]`, the solver plots solution components 1 and 3 as they are computed.

For the stiff solvers `ode15s`, `ode23s`, `ode23t`, and `ode23tb`, the Jacobian matrix $\partial f / \partial y$ is critical to reliability and efficiency. Use `odeset` to set `Jacobian` to `@FJAC` if `FJAC(T,Y)` returns the Jacobian $\partial f / \partial y$ or to the matrix $\partial f / \partial y$ if the Jacobian is constant. If the `Jacobian` property is not set (the default), $\partial f / \partial y$ is approximated by finite differences. Set the `Vectorized` property 'on' if the ODE function is coded so that `odefun(T,[Y1,Y2 ...])` returns [`odefun(T,Y1)`, `odefun(T,Y2)` `...`]. If $\partial f / \partial y$ is a sparse matrix, set the `JPattern` property to the sparsity pattern of $\partial f / \partial y$, i.e., a sparse matrix `S` with `S(i,j)` = 1 if the `i`th component of $f(t, y)$ depends on the `j`th component of y, and 0 otherwise.

The solvers of the ODE suite can solve problems of the form $M(t, y)y' = f(t, y)$, with time- and state-dependent mass matrix M. (The `ode23s` solver can solve only equations with constant mass matrices.) If a

problem has a mass matrix, create a function `M = MASS(t,y)` that returns the value of the mass matrix, and use `odeset` to set the `Mass` property to `@MASS`. If the mass matrix is constant, the matrix should be used as the value of the `Mass` property. Problems with state-dependent mass matrices are more difficult:

- If the mass matrix does not depend on the state variable y and the function `MASS` is to be called with one input argument, `t`, set the `MStateDependence` property to `'none'`.
- If the mass matrix depends weakly on y, set `MStateDependence` to `'weak'` (the default) and otherwise, to `'strong'`. In either case, the function `MASS` is called with the two arguments `(t,y)`.

If there are many differential equations, it is important to exploit sparsity:

- Return a sparse $M(t, y)$.
- Supply the sparsity pattern of $\partial f / \partial y$ using the `JPattern` property or a sparse $\partial f / \partial y$ using the `Jacobian` property.
- For strongly state-dependent $M(t, y)$, set `MvPattern` to a sparse matrix `S` with `S(i,j) = 1` if for any `k`, the `(i,k)` component of $M(t, y)$ depends on component `j` of y, and `0` otherwise.

If the mass matrix M is singular, then $M(t, y)y' = f(t, y)$ is a differential algebraic equation. DAEs have solutions only when y_0 is consistent, that is, if there is a vector yp_0 such that

$M(t_0, y_0)yp_0 = f(t_0, y_0)$. The `ode15s` and `ode23t` solvers can solve DAEs of index 1 provided that `y0` is sufficiently close to being consistent. If there is a mass matrix, you can use `odeset` to set the `MassSingular` property to `'yes'`, `'no'`, or `'maybe'`. The default value of `'maybe'` causes the solver to test whether the problem is a DAE. You can provide `yp0` as the value of the `InitialSlope` property. The default is the zero vector. If a problem is a DAE, and `y0` and `yp0` are not consistent, the solver treats them as guesses, attempts to compute consistent values that are close to the guesses, and continues to solve the problem. When solving DAEs, it is very advantageous to formulate the problem so that M is a diagonal matrix (a semi-explicit DAE).

Solver	Problem Type	Order of Accuracy	When to Use
`ode45`	Nonstiff	Medium	Most of the time. This should be the first solver you try.
`ode23`	Nonstiff	Low	If using crude error tolerances or solving moderately stiff problems.
`ode113`	Nonstiff	Low to high	If using stringent error tolerances or solving a computationally intensive ODE file.
`ode15s`	Stiff	Low to medium	If `ode45` is slow because the problem is stiff.
`ode23s`	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant.
`ode23t`	Moderately Stiff	Low	If the problem is only moderately stiff and you need a solution without numerical damping.
`ode23tb`	Stiff	Low	If using crude error tolerances to solve stiff systems.

The algorithms used in the ODE solvers vary according to order of accuracy [6] and the type of systems (stiff or nonstiff) they are designed to solve. See Algorithms for more details.

Options

Different solvers accept different parameters in the options list. For more information, see `odeset` and Improving ODE Solver Performance in the "Mathematics" section of the MATLAB documentation.

Parameters	ode45	ode23	ode113	ode15s	ode23s	ode23t	ode23tb
`RelTol, AbsTol, NormControl`	√	√	√	√	√	√	√
`OutputFcn, OutputSel, Refine, Stats`	√	√	√	√	√	√	√
`Events`	√	√	√	√	√	√	√
`MaxStep, InitialStep`	√	√	√	√	√	√	√
`Jacobian, JPattern, Vectorized`	--	--	--	√	√	√	√

`Mass`
`MStateDependence`
`MvPattern`
`MassSingular`
	√
	√
	--
	--
		√
		√
		--
		--
			√
			√
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							√
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							--

`InitialSlope`	--	--	--	√	--	√	--
`MaxOrder, BDF`	--	--	--	√	--	--	--

Examples

Example 1. An example of a nonstiff system is the system of equations describing the motion of a rigid body without external forces.

$$
\begin{aligned}
y'_1 &= y_2\, y_3 & y_1(0) &= 0 \\
y'_2 &= -y_1\, y_3 & y_2(0) &= 1 \\
y'_3 &= -0.51\, y_1\, y_2 & y_3(0) &= 1
\end{aligned}
$$

To simulate this system, create a function `rigid` containing the equations

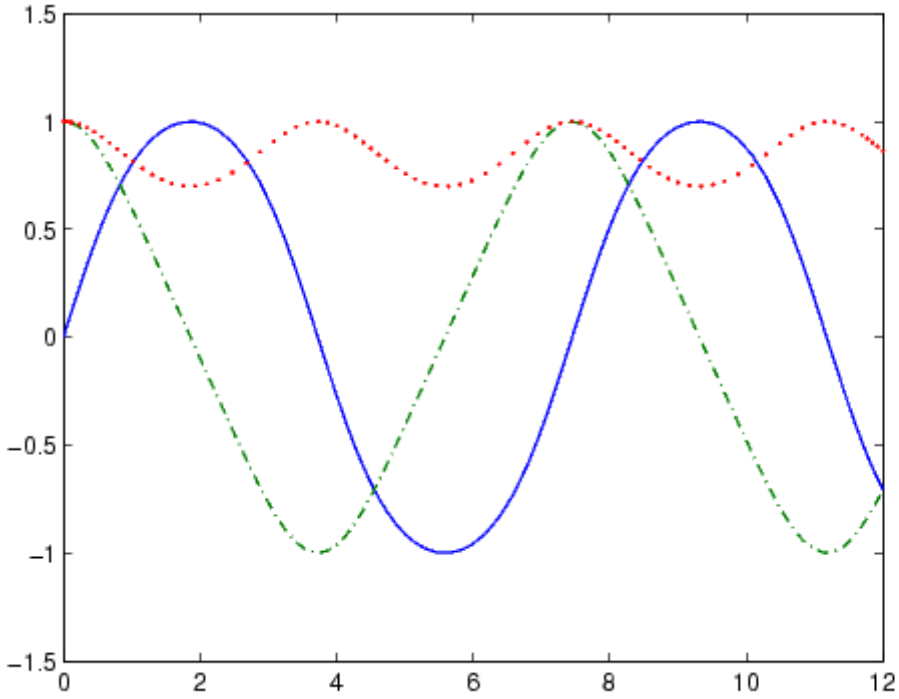
```
function dy = rigid(t,y)
dy = zeros(3,1);    % a column vector
dy(1) = y(2) * y(3);
dy(2) = -y(1) * y(3);
dy(3) = -0.51 * y(1) * y(2);
```

In this example we change the error tolerances using the `odeset` command and solve on a time interval `[0 12]` with an initial condition vector `[0 1 1]` at time `0`.

```
options = odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-5]);
[T,Y] = ode45(@rigid,[0 12],[0 1 1],options);
```

Plotting the columns of the returned array `Y` versus `T` shows the solution

```
plot(T,Y(:,1),'-',T,Y(:,2),'-.',T,Y(:,3),'.')
```



Example 2. An example of a stiff system is provided by the van der Pol equations in relaxation oscillation. The limit cycle has portions where the solution components change slowly and the problem is quite stiff, alternating with regions of very sharp change where it is not stiff.

$$y'_1 = y_2 \qquad y_1(0) = 0$$
$$y'_2 = 1000(1 - y_1^2)y_2 - y_1 \qquad y_2(0) = 1$$

To simulate this system, create a function `vdp1000` containing the equations

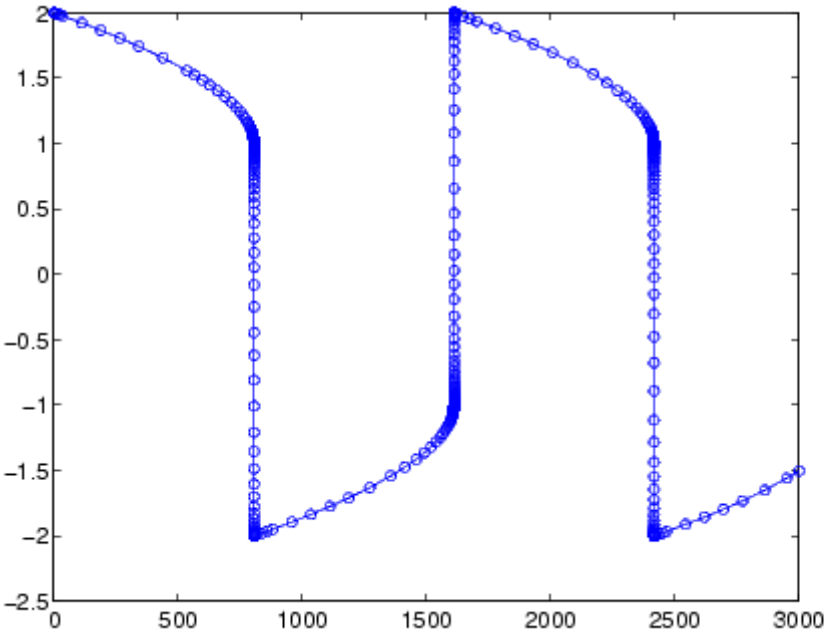
```
function dy = vdp1000(t,y)
dy = zeros(2,1);    % a column vector
dy(1) = y(2);
dy(2) = 1000*(1 - y(1)^2)*y(2) - y(1);
```

For this problem, we will use the default relative and absolute tolerances (`1e-3` and `1e-6`, respectively) and solve on a time interval of `[0 3000]` with initial condition vector `[2 0]` at time `0`.

```
[T,Y] = ode15s(@vdp1000,[0 3000],[2 0]);
```

Plotting the first column of the returned matrix `Y` versus `T` shows the solution

```
plot(T,Y(:,1),'-o')
```



Algorithms

`ode45` is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair. It is a *one-step* solver - in computing `y(t`$_n$`)`, it needs only the solution at the immediately preceding time point, `y(t`$_{n-1}$`)`. In general, `ode45` is the best function to apply as a "first try" for most problems. [3]

`ode23` is an implementation of an explicit Runge-Kutta (2,3) pair of Bogacki and Shampine. It may be more efficient than `ode45` at crude tolerances and in the presence of moderate stiffness. Like `ode45`, `ode23` is a one-step solver. [2]

`ode113` is a variable order Adams-Bashforth-Moulton PECE solver. It may be more efficient than `ode45` at stringent tolerances and when the ODE file function is particularly expensive to evaluate. `ode113` is a *multistep* solver - it normally needs the solutions at several preceding time points to compute the current solution. [7]

The above algorithms are intended to solve nonstiff systems. If they appear to be unduly slow, try using one of the stiff solvers below.

`ode15s` is a variable order solver based on the numerical differentiation formulas (NDFs). Optionally, it uses the backward differentiation formulas (BDFs, also known as Gear's method) that are usually less efficient. Like `ode113`, `ode15s` is a multistep solver. Try `ode15s` when `ode45` fails, or is very inefficient, and you suspect that the problem is stiff, or when solving a differential-algebraic problem. [9], [10]

`ode23s` is based on a modified Rosenbrock formula of order 2. Because it is a one-step solver, it may be more efficient than `ode15s` at crude tolerances. It can solve some kinds of stiff problems for which `ode15s` is not effective. [9]

`ode23t` is an implementation of the trapezoidal rule using a "free" interpolant. Use this solver if the problem is only moderately stiff and you need a solution without numerical damping. `ode23t` can solve DAEs. [10]

`ode23tb` is an implementation of TR-BDF2, an implicit Runge-Kutta formula with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two. By construction, the same iteration matrix is used in evaluating both stages. Like `ode23s`, this solver may be more efficient than `ode15s` at crude tolerances. [8], [1]

See Also

deval, odeset, odeget, @ (function handle)

References

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[2] Bogacki, P. and L. F. Shampine, "A 3(2) pair of Runge-Kutta formulas," *Appl. Math. Letters*, Vol. 2, 1989, pp 1-9.

[3] Dormand, J. R. and P. J. Prince, "A family of embedded Runge-Kutta formulae," *J. Comp. Appl. Math.*, Vol. 6, 1980, pp 19-26.

[4] Forsythe, G. , M. Malcolm, and C. Moler, *Computer Methods for Mathematical Computations*, Prentice-Hall, New Jersey, 1977.

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[6] Shampine, L. F. , *Numerical Solution of Ordinary Differential Equations*, Chapman & Hall, New York, 1994.

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[10] Shampine, L. F., M. W. Reichelt, and J.A. Kierzenka, "Solving Index-1 DAEs in MATLAB and Simulink," *SIAM Review*, Vol. 41, 1999, pp 538-552.