LU decomposition Method

Gauss elimination becomes inefficient when solving equations with the same coefficients for [A] but with different b's.
LU decomposition separates the time consuming elimination of [A] form the manipulation of $\{b\}$. Hence, the decomposed [A] could be used with several $\{b\}$'s in an efficient manner.

LU decomposition is based on the fact that any square matrix [A] can be written as a product of two matrices as:

[A]=[L][U]

Where [L] is a lower triangular matrix and [U] is an upper triangular matrix.

Crout's method

To illustrate the Crout's method for LU decomposition, consider the following 3×3 matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Hence

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & (l_{11}u_{12}) & (l_{11}u_{13}) \\ l_{21} & (l_{21}u_{12}+l_{22}) & (l_{21}u_{13}+l_{22}u_{23}) \\ l_{31} & (l_{31}u_{12}+l_{32}) & (l_{31}u_{13}+l_{32}u_{23}+l_{33}) \end{bmatrix}$$

Elements of the matrices [L] and [U] can be found by equating the two above matrices:

$$\begin{cases} l_{11} = a_{11}; l_{21} = a_{21}; l_{31} = a_{31} \\ l_{11}u_{12} = a_{12}, hence\ u_{12} = \dfrac{a_{12}}{l_{11}} = \dfrac{a_{12}}{a_{11}} \\ l_{21}u_{12} + l_{22} = a_{22}, hence\ l_{22} = a_{22} - l_{21}u_{12} \\ l_{31}u_{12} + l_{32} = a_{32}, hence\ l_{32} = a_{32} - l_{31}u_{12} \\ l_{11}u_{13} = a_{13}, hence\ u_{13} = \dfrac{a_{13}}{l_{11}} = \dfrac{a_{13}}{a_{11}} \\ l_{21}u_{13} + l_{22}u_{23} = a_{23}, hence\ u_{23} = \dfrac{a_{23} - l_{21}u_{13}}{l_{22}} \\ l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}, hence\ l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \end{cases}$$

For a general n×n matrix, the following expressions can be applied to find the LU decomposition of a matrix [A]:

$$l_{ij} = \left\{ a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right\};\ i \geq j;\ i = \mathbf{1}, 2, \ldots, \mathbf{n}$$

$$u_{ij} = \left\{ \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}} \right\};\ i < j;\ j = \mathbf{2}, 3, \ldots, \mathbf{n}$$

and $u_{ii} = 1$; i=1,2,…,n

Note: It is better to follow a certain order when computing the terms of the [L] and [U] matrices. This order is: l_{i1}, u_{1j}; l_{i2}, u_{2j}; …; $l_{i,n-1}, u_{n-1,j}$; l_{nn}.

***Properties of* LU *matrices of* A=LU**

(a) The **L** matrices: all 1's in diagonal, same multipliers l_{ij} as in the elimination matrices
(b) When a row of **A** starts with zeros, so does that row of **L**
(c) When a column of **A** starts with zeros, so does that column of **U**

Example

Find the LU decomposition of the matrix

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

$$[A] = [L][U]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The $[U]$ matrix is the same as found at the end of the forward elimination of Naïve Gauss elimination method, that is

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

To find ℓ_{21} and ℓ_{31}, find the multiplier that was used to make the a_{21} and a_{31} elements zero in the first step of forward elimination of the Naïve Gauss elimination method. It was

$$\ell_{21} = \frac{64}{25}$$
$$= 2.56$$

$$\ell_{31} = \frac{144}{25}$$
$$= 5.76$$

To find ℓ_{32}, what multiplier was used to make a_{32} element zero? Remember a_{32} element was made zero in the second step of forward elimination. The $[A]$ matrix at the beginning of the second step of forward elimination was

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

So

$$\ell_{32} = \frac{-16.8}{-4.8}$$
$$= 3.5$$

Hence

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Confirm $[L][U] = [A]$.

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$
$$= \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Another example

Perform LU decomposition of the following matrix

10	2	1
-3	-6	2
1	1	5

Multiply the first row by f_{21} = -3/10 = -0.3 and subtract the result from the second row to eliminate the a_{21} term. Then, multiply the first row by f_{31} = 1/10 = 0.1 and subtract the result from the third row to eliminate the a_{31} term. The result is

10	2	1
0	5.4	1.7
0	0.8	5.1

Multiply the second row by f_{32} = 0.8/(-5.4) = -0.148148 and subtract the result from the third row to eliminate the a_{32} term.

10	2	1
0	5.4	1.7
0	0	5.351852

Therefore, the LU decomposition is

$$[L][U]=\begin{bmatrix}1 & 0 & 0\\ -0.3 & 1 & 0\\ 0.1 & -0.148148 & 1\end{bmatrix}\begin{bmatrix}10 & 2 & -1\\ 0 & -5.4 & 1.7\\ 0 & 0 & 5.351852\end{bmatrix}$$

Multiplying [L] and [U] yields the original matrix as verified by the following MATLAB session,

```
>> L = [1 0 0;-0.3 1 0;0.1 -0.148148 1];
>> U = [10 2 -1;0 -5.4 1.7; 0 0 5.351852];
>> A = L*U
A =
10.0000          2.0000  -1.0000
-3.0000 -6.0000 2.0000
1.0000  1.0000  5.0000
```

Solution of linear equations by LU decomposition

Now to solve the system of linear equations, the initial system can be expressed as:

$$[A]\{x\}=\{b\}$$

Under the following form

$$[A]\{x\}=[L][U]\{x\}=\{b\}$$

To find the solution $\{x\}$, the first a vector $\{z\}$ can be defined:

$$\{z\}=[U]\{x\}$$

Our initial system becomes, then: $[L]\{z\}=\{b\}$

As [L] is a lower triangular matrix the $\{z\}$ can be computed starting by z_1 until z_n. Then the values of $\{x\}$ can be found using the equation:

$$\{z\}=[U]\{x\}$$

as [U] is an upper triangular matrix, it is possible to compute $\{x\}$ using a back substitution process starting x_n until x_1. [You will better understand with an example …]

The general form to solve a system of linear equations using LU decomposition is:

$$z_1 = \frac{b_1}{l_{11}}$$

$$z_i = \frac{b_i - \sum_{k=1}^{i-1} l_{ik} z_k}{l_{ii}}; \; i = 2,3,...,n$$

And

$$x_n = z_n$$

$$x_i = z_i - \sum_{k=i+1}^{n} u_{ik} x_k; \; i = n-1, n-2,...,2,1$$

Example

Use the LU decomposition method to solve the following simultaneous linear equations.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solution

Recall that

$$[A][X] = [C]$$

and if

$$[A] = [L][U]$$

then first solving

$$[L][Z] = [C]$$

and then

$$[U][X] = [Z]$$

gives the solution vector $[X]$.

Now in the previous example, we showed

$$[A] = [L][U]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

First solve

$$[L][Z] = [C]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

to give

$$z_1 = 106.8$$

$$2.56 z_1 + z_2 = 177.2$$

$$5.76z_1 + 3.5z_2 + z_3 = 279.2$$

Forward substitution starting from the first equation gives

$$z_1 = 106.8$$

$$\begin{aligned} z_2 &= 177.2 - 2.56z_1 \\ &= 177.2 - 2.56 \times 106.8 \\ &= -96.208 \end{aligned}$$

$$\begin{aligned} z_3 &= 279.2 - 5.76z_1 - 3.5z_2 \\ &= 279.2 - 5.76 \times 106.8 - 3.5 \times (-96.208) = 0.76 \end{aligned}$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

This matrix is same as the right hand side obtained at the end of the forward elimination steps of Naïve Gauss elimination method. Is this a coincidence?

Now solve

$$[U][X] = [Z]$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$0.7a_3 = 0.76$$

From the third equation

$$0.7a_3 = 0.76$$

$$\begin{aligned} a_3 &= \frac{0.76}{0.7} \\ &= 1.0857 \end{aligned}$$

Substituting the value of a_3 in the second equation,

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$\begin{aligned} a_2 &= \frac{-96.208 + 1.56a_3}{-4.8} \\ &= \frac{-96.208 + 1.56 \times 1.0857}{-4.8} \\ &= 19.691 \end{aligned}$$

Substituting the value of a_2 and a_3 in the first equation,

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5 \times 19.691 - 1.0857}{25}$$

$$= 0.29048$$

Hence the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.691 \\ 1.0857 \end{bmatrix}$$

Inverse of a square matrix using LU decomposition?

A matrix $[B]$ is the inverse of $[A]$ if

$$[A][B] = [I] = [B][A].$$

How can we use LU decomposition to find the inverse of the matrix? Assume the first column of $[B]$ (the inverse of $[A]$) is

$[b_{11} \, b_{12} \ldots \ldots \, b_{n1}]^{\mathrm{T}}$

Then from the above definition of an inverse and the definition of matrix multiplication

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly the second column of $[B]$ is given by

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly, all columns of $[B]$ can be found by solving n different sets of equations with the column of the right hand side being the n columns of the identity matrix.

Example

Use LU decomposition to find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

Knowing that

$$[A] = [L][U]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

We can solve for the first column of $[B] = [A]^{-1}$ by solving for

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

First solve

$$[L][Z] = [C],$$

that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

to give

$$z_1 = 1$$
$$2.56 z_1 + z_2 = 0$$
$$5.76 z_1 + 3.5 z_2 + z_3 = 0$$

Forward substitution starting from the first equation gives

$$z_1 = 1$$
$$\begin{aligned} z_2 &= 0 - 2.56 z_1 \\ &= 0 - 2.56(1) \\ &= -2.56 \end{aligned}$$
$$\begin{aligned} z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\ &= 0 - 5.76(1) - 3.5(-2.56) \\ &= 3.2 \end{aligned}$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Now solve

$$[U][X] = [Z]$$

that is

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$
$$-4.8b_{21} - 1.56b_{31} = -2.56$$
$$0.7b_{31} = 3.2$$

Backward substitution starting from the third equation gives

$$\begin{aligned}
b_{31} &= \frac{3.2}{0.7} \\
&= 4.571 \\
b_{21} &= \frac{-2.56 + 1.56b_{31}}{-4.8} \\
&= \frac{-2.56 + 1.56(4.571)}{-4.8} \\
&= -0.9524 \\
b_{11} &= \frac{1 - 5b_{21} - b_{31}}{25} \\
&= \frac{1 - 5(-0.9524) - 4.571}{25} \\
&= 0.04762
\end{aligned}$$

Hence the first column of the inverse of $[A]$ is

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Similarly by solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ gives } \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

and solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ gives } \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Hence

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$