Linear Algebraic Equations

- Matrix Inversion / Gauss Elimination
- LU decomposition (product of a lower and an upper triangular matrix)
- Jacobi and Gauss-Seidel methods (the Liebmann method or the method of successive displacement, an iterative method used to solve a linear system of equations)
- Computer-based solutions
  - LSARG subroutine (Fortran IMSL subroutine that solves a real general system of linear equations with iterative refinement)
  - MATLAB solution of system of linear equations (backslash operator \ , left division, rref function)

Linear Algebraic Equations

- Review of Matrix Form Representation of System of Linear Equations

What is a matrix?

- A **matrix** is a rectangular array of **elements**

$$\begin{bmatrix} -5 & 0 & 1 & 2 \\ 3 & -4 & -9 & 2 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

- The elements may be of any type (e.g. integer, real, complex, logical, or even other matrices).
- Matrices with integer, real, or complex elements will be considered.

Order of matrices...

- Order 4×3:

$$\begin{array}{cc} & 3\ \ columns \rightarrow \\ \begin{array}{c} 4\ \ rows \\ \downarrow \end{array} & \begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} \end{array}$$

- Order 3×4:

$$\begin{array}{cc} & 4\ \ columns \rightarrow \\ \begin{array}{c} 3\ \ rows \\ \downarrow \end{array} & \begin{bmatrix} -5 & 0 & 1 & 2 \\ 3 & -4 & -9 & 2 \\ 3 & 1 & 4 & 2 \end{bmatrix} \end{array}$$

…Order of matrices

- Order 2 × 4:

$$\begin{array}{cc} & 4 \;\; columns \rightarrow \\ \begin{array}{c} 2 \;\; rows \\ \downarrow \end{array} & \begin{bmatrix} -23 & -0.5 & 4.3 & 12 \\ 8 & 2 & 8 & 1 \end{bmatrix} \end{array}$$

- Order 1 × 6:

$$\begin{array}{cc} & 6 \;\; columns \rightarrow \\ \begin{array}{c} 1 \;\; rows \\ \downarrow \end{array} & \begin{bmatrix} 2 & 1 & 1 & -2 & 1 & -5 \end{bmatrix} \end{array}$$

- Order 3 × 1:

$$\begin{array}{cc} & 1 \;\; columns \rightarrow \\ \begin{array}{c} 3 \;\; rows \\ \downarrow \end{array} & \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \end{array}$$

Specifying matrix elements

$$\mathbf{A} = \begin{array}{cc} & column \;\; j \rightarrow \\ \begin{array}{c} row \;\; i \\ \downarrow \end{array} & \begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} \end{array}$$

- a_{ij} denotes the element of the matrix **A** on the i^{th} row and j^{th} column.

- a_{12} = 0
- a_{21} = 2
- a_{23} = -4
- a_{32} = 2
- a_{41} = 3
- a_{43} = 4

What are matrices used for?

- Transformations
- Transitions
- Linear equations

Matrix operations: scalar multiplication

- Multiplying an $m \times n$ matrix by a scalar results in an $m \times n$ matrix with each of its elements multiplied by the scalar.
- e.g.

$$3\begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 3 \\ 6 & 9 & -12 \\ -27 & 6 & 18 \\ 9 & 3 & 12 \end{bmatrix}$$

$$-2\begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 & -2 \\ -4 & -6 & 8 \\ 18 & -4 & -12 \\ -6 & -2 & -8 \end{bmatrix}$$

Matrix operations: addition…

- Adding or subtracting an $m \times n$ matrix by an $m \times n$ matrix results in an $m \times n$ matrix with each of its elements added or subtracted.
- e.g.

$$B = \begin{bmatrix} -4 & -7 \\ -9 & -9 \\ 5 & -2 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -5 \\ 7 & x \\ -y & 2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} -4+2 & -7+(-5) \\ -9+7 & -9+x \\ 5+(-y) & -2+2 \end{bmatrix}$$

$$A+B \Rightarrow \begin{bmatrix} -2. & -12. \\ -2. & x-9. \\ 5.-y & 0. \end{bmatrix}$$

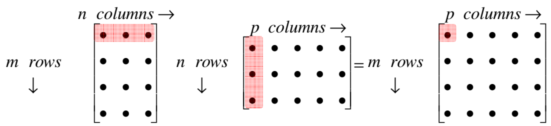
…Matrix operations: addition

- Note that matrices being added or subtracted must be of the same order.
- e.g.

$$\begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \textit{invalid!}$$

Matrix operations: multiplication…

- Multiplying an $m \times n$ matrix by an $n \times p$ matrix results in an $m \times p$ matrix



…Matrix operations: multiplication…

- Example 1…

$$\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & \\ & \end{bmatrix}$$

$$(-1 \times 2) + (0 \times 3) + (2 \times 1) = 0$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ & \end{bmatrix}$$

$$(-1 \times -1) + (0 \times -2) + (2 \times -4) = -7$$

…Matrix operations: multiplication…

- …Example 1

$$\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & -4 \end{bmatrix}=\begin{bmatrix} 0 & -7 \\ 10 & \end{bmatrix}$$

$$(3\times 2)+(1\times 3)+(1\times 1)=10$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & -4 \end{bmatrix}=\begin{bmatrix} 0 & -7 \\ 10 & -9 \end{bmatrix}$$

$$(3\times -1)+(1\times -2)+(1\times -4)=-9$$

…Matrix operations: multiplication…

- Example 2…

$$\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 3 & \\ & \end{bmatrix}$$

$$(2\times 3)+(-3\times 1)=3$$

$$\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 3 & -8 \\ & \end{bmatrix}$$

$$(2\times -1)+(-3\times 2)=-8$$

…Matrix operations: multiplication…

- …Example 2

$$\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 3 & -8 \\ -1 & \end{bmatrix}$$

$$(1\times 3)+(-4\times 1)=-1$$

$$\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 3 & -8 \\ -1 & -9 \end{bmatrix}$$

$$(1\times -1)+(-4\times 2)=-9$$

…Matrix operations: multiplication…

- Example 3

$$\begin{bmatrix} 1 & -5 & 3 \\ -2 & 6 & 8 \\ 4 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}=\textbf{\textit{invalid!}}$$

- i.e. the number of columns in the first matrix must equal the number of rows in the second matrix!

…Matrix operations: multiplication…

- Example 4

$$\begin{bmatrix}4 & -2\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}-6\end{bmatrix}$$

$$(4\times 1)+(-2\times 5)=-6$$

…Matrix operations: multiplication…

- Example 5…

$$\begin{bmatrix}1\\5\end{bmatrix}\begin{bmatrix}4 & -2\end{bmatrix}=\begin{bmatrix}4 & \\ & \end{bmatrix}$$

$$(1\times 4)=4$$

$$\begin{bmatrix}1\\5\end{bmatrix}\begin{bmatrix}4 & -2\end{bmatrix}=\begin{bmatrix}4 & -2\\ & \end{bmatrix}$$

$$(1\times -2)=-2$$

…Matrix operations: multiplication…

- …Example 5

$$\begin{bmatrix}1\\5\end{bmatrix}\begin{bmatrix}4 & -2\end{bmatrix}=\begin{bmatrix}4 & -2\\20 & \end{bmatrix}$$

$$(5\times 4)=20$$

$$\begin{bmatrix}1\\5\end{bmatrix}\begin{bmatrix}4 & -2\end{bmatrix}=\begin{bmatrix}4 & -2\\20 & -10\end{bmatrix}$$

$$(5\times -2)=-10$$

…Matrix operations: multiplication

- Matrix multiplication is NOT commutative
- In general, if **A** and **B** are two matrices then **A B ≠ B A**
- i.e. the order of matrix multiplication is important!
- e.g.

$$\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=\begin{bmatrix}2 & 4\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}2 & 2\\0 & 1\end{bmatrix}$$

Matrix operations: transpose…

- If **B** = **A**^{T}, then $b_{ij} = a_{ji}$
- i.e. the transpose of an $m \times n$ matrix is an $n \times m$ matrix with the rows and columns swapped.
- e.g.

$$\begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix}^T = \begin{bmatrix} -5 & 2 & -9 & 3 \\ 0 & 3 & 2 & 1 \\ 1 & -4 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ 2 \\ -9 \\ 3 \end{bmatrix}^T = \begin{bmatrix} -5 & 2 & -9 & 3 \end{bmatrix}$$

…Matrix operations: transpose

- $(\mathbf{A}\ \mathbf{B})^T = \mathbf{B}^T\ \mathbf{A}^T$
- Note the reversal of order.
- Justification (*not a proof*):
  e.g. if **A** is 3×2 and **B** is 2×4
  then **A**^{T} is 2×3 and **B**^{T} is 4×2
  so **A**^{T} **B**^{T} cannot be multiplied
  but **B**^{T} **A**^{T} can be multiplied.

Special matrices: row and column

- A $1 \times$ n matrix is called a **row** matrix.
  e.g.

$$\begin{array}{cc} & 6 \ \ columns \rightarrow \\ \begin{array}{c} 1 \ \ rows \\ \downarrow \end{array} & \begin{bmatrix} 2 & 1 & 1 & -2 & 1 & -5 \end{bmatrix} \end{array}$$

- An m $\times$ 1 matrix is called a **column** matrix.
  e.g.

$$\begin{array}{cc} & 1 \ \ columns \rightarrow \\ \begin{array}{c} 3 \ \ rows \\ \downarrow \end{array} & \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \end{array}$$

Special matrices: square

- An n $\times$ n matrix is called a **square** matrix.
  i.e. a square matrix has the same number of rows and columns.
  e.g.

$$\begin{bmatrix} 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 5 \\ -3 & 2 & -5 & 0 \\ 0 & -5 & 3 & -7 \\ 5 & 0 & 7 & 4 \end{bmatrix}$$

Special matrices: diagonal

- A square matrix is **diagonal** if non-zero elements only occur on the leading diagonal.
  i.e. $a_{ij} = 0$ for $i \neq j$
  e.g.

$$[1] \qquad \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

- Premultiplying a matrix by a diagonal matrix scales each row by the diagonal element.
- Postmultiplying a matrix by a diagonal matrix scales each column by the diagonal element.

Special matrices: triangular

- **A lower triangular** matrix is a square matrix having all elements above the leading diagonal zero.
  e.g.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & -5 & 3 & 0 \\ 5 & 0 & 7 & 4 \end{bmatrix}$$

- **An upper triangular** matrix is a square matrix having all elements below the leading diagonal zero.
  e.g.

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Special matrices: null

- The **null** matrix, **0**, behaves like 0 in arithmetic addition and subtraction.
- Null matrices can be of any order and have all of their elements zero.

$$\mathbf{0} = [0] \qquad \mathbf{0} = [0 \quad 0 \quad 0]$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Special matrices: identity…

- The **identity** matrix, **I**, behaves like 1 in arithmetic multiplication.
- Identity matrices are diagonal. They have 1s on the diagonal and 0s elsewhere.
  e.g.

$$\mathbf{I} = [1] \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- In the world of the matrix the identity truly is ‘the one’.

...Special matrices: identity

- The identity matrix multiplied by any compatible matrix results in the same matrix.
  i.e. **I A = A**
  e.g. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}=\begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}$
- Any matrix multiplied by a compatible identity matrix results in the same matrix.
  i.e. **A I = A**
  e.g. $\begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}$
- Multiplication by the identity matrix is thus commutative.

Determinant of a 2×2 matrix...

- Matrices can represent geometric transformations, such as scaling, rotation, shear, and mirroring.
- 2×2 matrices can represent geometric transformations in a 2–dimensional space, such as a plane.
- Determinants of 2×2 matrices give us information about how such transformations change the area of shapes.
- Determinants are also useful to define the inverse of a matrix.

• In Matlab: ***det(A)*** = **det(A)**

Determinant of a 2×2 and 3x3 matrices...

- To find the determinant of a 2 x 2 matrix, multiply diagonal #1 and subtract the product of diagonal #2.
- To find the determinant of a 3 x 3 matrix, first recopy the first two columns. Then do 6 diagonal products.

  The determinant of the matrix is the sum of the downwards products minus the sum of the upwards products

Diagonal 2 = -2

$$\begin{vmatrix}3 & -1\\2 & 4\end{vmatrix} = 12-(-2)=14$$

Diagonal 1 = 12

18 60 16

$$\begin{vmatrix}5 & 2 & 6\\2 & -1 & 4\\-3 & 3 & 4\end{vmatrix}\begin{matrix}5 & 2\\2 & -1\\-3 & 3\end{matrix}$$

-20 -24 36

= (-8) - (94) = **-102**

...Determinant of a 2×2 matrix...

- The determinant of a 2×2 matrix is the product of the 2 leading diagonal terms minus the product of the cross- diagonal.
- i.e. if **A** is a 2×2 matrix, then the determinant of **A** is denoted by det(**A**) = |**A**| = $a_{11}\, a_{22} - a_{21}\, a_{12}$
- e.g.

$$\det\begin{pmatrix}3 & -1\\2 & 6\end{pmatrix}=\begin{vmatrix}3 & -1\\2 & 6\end{vmatrix}=(3\times 6)-(2\times -1)=20$$

$$\det\begin{pmatrix}-2 & 5\\1 & 3\end{pmatrix}=\begin{vmatrix}-2 & 5\\1 & 3\end{vmatrix}=(-2\times 3)-(1\times 5)=-11$$

…Determinant of a 2×2 matrix

- |**A B**| = |**A**| |**B**|
- Note the order is not important.
- Justification (*not a proof*):
  We will shortly see that **A** and **B** can represent geometric transformations, and **A B** represents the combined transformation of **B** followed by **A**. The determinant represents the factor by which the area is changed, so the combined transformation changes area by a factor |**A B**|. Looking at the individual transformations, the area of the first is changed by a factor |**B**|, and the second by |**A**|. The overall transformation is thus changed by a factor |**B**| |**A**|, which is the same as |**A**| |**B**|.

Inverse of a matrix

- In arithmetic multiplication the inverse of a number c is $1/c$ since
  $c \times 1/c = 1$ and $1/c \times c = 1$
- For matrices the inverse of a matrix **A** is denoted by $\mathbf{A}^{-1}$
- $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$
  $\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$
  where **I** is the identity matrix.
- Multiplication of a matrix by its inverse is thus commutative.
- the inverse of 2 × 2 matrices will be considered.

Inverse of a 2×2 matrix…

- The inverse of a 2 × 2 matrix **A** is given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- Note:
  The leading term is 1/determinant;
  The diagonal elements are swapped;
  The cross-diagonal elements change their sign.

Inverse of a 2×2 matrix…

- Step 1: Exchange the elements in the leading diagonal.
- Step 2: Change the sign of the other two elements.
- Step 3: Multiply by the reciprocal of the determinant

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

…Inverse of a 2×2 matrix…

- Example 1

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix}} \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

- Note that **A A**$^{-1}$ = **I** (right inverse)

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- and **A**$^{-1}$ **A** = **I** (left inverse)

$$\frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

…Inverse of a 2×2 matrix…

- Example 2

$$\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix}} \begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0.3 & -0.2 \\ 0.2 & 0.2 \end{bmatrix}$$

- Note that **A A**$^{-1}$ = **I** (right inverse)

$$\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0.3 & -0.2 \\ 0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- and **A**$^{-1}$ **A** = **I** (left inverse)

$$\begin{bmatrix} 0.3 & -0.2 \\ 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

…Inverse of a 2×2 matrix…

- Example 3

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}} \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} = \frac{1}{0}\begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} = \textit{invalid!}$$

- Note that the determinant is zero so the inverse does not exist for this matrix.
- Matrices with zero determinant can have no inverse.
- Such matrices are called **singular**.

• A matrix A has an inverse matrix A^{-1} if and only if **det(A)≠0.**

…Inverse of a 2×2 matrix

- (**A B**)$^{-1}$ = **B**$^{-1}$ **A**$^{-1}$
- Note the reversal of order.
- Justification (*not a proof*):
  B^{-1} **A**$^{-1}$ **A B** = **B**$^{-1}$ (**A**$^{-1}$ **A**) **B** = **B**$^{-1}$ **B** = **I**
  so **B**$^{-1}$ **A**$^{-1}$ is the inverse of **A B**
  i.e. (**A B**)$^{-1}$ = **B**$^{-1}$ **A**$^{-1}$

Eigenvalues and Eigenvectors

- This is a way of rearranging and consolidating the variance in a matrix

$$\underset{MxM}{D}\ \underset{Mx1}{V} = \underset{1x1}{\lambda}\ \underset{Mx1}{V}$$

$$D = any\ square\ matrix$$

$$V = Eigenvector$$

$$\lambda = Eigenvalue$$

- Think of it as taking a matrix and allowing it to be represented by a scalar and a vector (actually a few scalars and vectors, because there is usually more than one solution).

Eigenvalues and Eigenvectors

- Another way to look at it is that we are trying to come up with λ and V that will allow for the following equation to be true

$$(D-\lambda I)V = 0$$

- D is the matrix of interest, I the identity matrix

$$(D-\lambda I)V = 0$$

$$\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Eigenvalues and Eigenvectors

If v1 and v2 equal zero the previous statement is true, but uninteresting.

A non-trivial solution is found when the determinate of the leftmost matrix is set to be 0.

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

Solving for the eigenvalues λ requires solving for the roots of this polynomial

Generalize it to $x\lambda^2 - y\lambda + z = 0$

To solve for λ apply:

$$\lambda = \frac{-y \pm \sqrt{y^2 - 4xz}}{2x}$$

Eigenvalues and Eigenvectors

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda^2 - (5+2)\lambda + 5*2 - 1*4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda = \frac{-(-7) + \sqrt{7^2 - 4*1*6}}{2*1} = 6$$

$$\lambda = \frac{-(-7) - \sqrt{7^2 - 4*1*6}}{2*1} = 1$$

$$\lambda_1 = 6,\ \lambda_2 = 1$$

Eigenvalues and Eigenvectors

- Using the first eigenvalue its corresponding eigenvector is found

$$\begin{bmatrix} 5-6 & 1 \\ 4 & 2-6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

This gives you two equations:

$$-1v_1 + 1v_2 = 0$$

$$4v_1 - 4v_2 = 0$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\lambda_1 = 6, \lambda_2 = 1$$

Eigenvalues and Eigenvectors

- Using the second eigenvalue its corresponding eigenvector is found

$$\begin{bmatrix} 5-1 & 1 \\ 4 & 2-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

This gives you two equations:

$$4v_1 + 1v_2 = 0$$

$$4v_1 + 1v_2 = 0$$

$$V_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\lambda_1 = 6, \lambda_2 = 1$$

Eigenvalues and Eigenvectors

- Let's show that the original equation holds

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \text{ and } 6 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \text{ and } 1 * \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$(D - \lambda I)V = 0$$

$$\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\lambda_1 = 6, \lambda_2 = 1$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Matrix in Matlab

```
>>
>> C=[1 2 3;4 5 6]

C =

     1     2     3
     4     5     6
```

```
>> A=[7 2 4; 63 8 3; 9 8 1]

A =

     7     2     4
    63     8     3
     9     8     1
```

```
>> B=[7 2 4;
63 8 3;
9 8 1]

B =

     7     2     4
    63     8     3
     9     8     1
```

Matrix in Matlab

```
>>
>> ov=ones(1, 4)

ov =

     1     1     1     1

>>
>> om=ones(3,3)

om =

     1     1     1
     1     1     1
     1     1     1
```

```
>>
>> zv=zeros(1, 4)

zv =

     0     0     0     0

>>
>> zm=zeros(3,3)

zm =

     0     0     0
     0     0     0
     0     0     0
```


Matrix in Matlab

- Addition

```
>> A=eye(3,3)

A =

     1     0     0
     0     1     0
     0     0     1
```

```
>> B=ones(3,3)

B =

     1     1     1
     1     1     1
     1     1     1
```

```
>> C=A+B

C =

     2     1     1
     1     2     1
     1     1     2
```

- Multiplication

```
>> A=[1 2;3 4]

A =

     1     2
     3     4
```

```
>> B=[1 1; 2 2]

B =

     1     1
     2     2
```

```
>> C=A*B

C =

     5     5
    11    11
```

- Inverse

```
>> A=[1 3 1; 1 2 1; 1 5 0]

A =

     1     3     1
     1     2     1
     1     5     0
```

```
>> det(A)

ans =

     1
```

```
>> A^(-1)

ans =

    -5     5     1
     1    -1     0
     3    -2    -1
```

```
>> inv(A)

ans =

    -5     5     1
     1    -1     0
     3    -2    -1
```


Matrix in Matlab

- X=matrix
- ;=end of a row
- :=all row or column

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Subscripting – each element of a matrix can be addressed with a pair of numbers; row first, column second

X(2,3) **6**

X(3, :) = $(7 \quad 8 \quad 9)$

X([2 3], 2) = $\begin{pmatrix} 5 \\ 8 \end{pmatrix}$

"Special" matrix commands:

- **zeros(3,1)** = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- **ones(2)** = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
- **magic(3)** = $\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$

Matrix in Matlab

- Product **A*B**
  If either factor is 1X1, i.e., a scalar, then this is scalar multiplication.
- Inverse **A^(-1)** or **inv(A)**
  There is also a pseudoinverse, **pinv**, *for nonsquare matrices.*
- Determinant **det(A)**