Linear Algebraic Equations

- Matrix Inversion / Gauss Elimination
- LU decomposition (product of a lower and an upper triangular matrix)
- Jacobi and Gauss-Seidel methods (the Liebmann method or the method of successive displacement, an iterative method used to solve a linear system of equations)
- Computer-based solutions
  - LSARG subroutine (Fortran IMSL subroutine that solves a real general system of linear equations with iterative refinement)
  - MATLAB solution of system of linear equations (backslash operator \ , left division, rref function)

What is a matrix?

- A **matrix** is a rectangular array of **elements**

$$\begin{bmatrix} -5 & 0 & 1 & 2 \\ 3 & -4 & -9 & 2 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

- The elements may be of any type (e.g. integer, real, complex, logical, or even other matrices).

Linear equations…

- A set of *n* linear equations in *n* variables, x_i

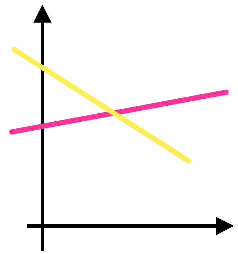
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

- can be written in matrix form, **Ax** = **b**:

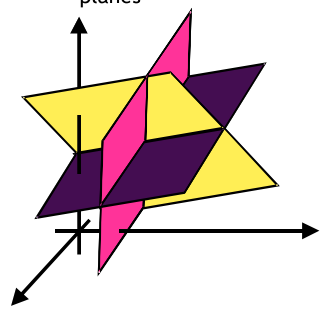
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Visualizing $n \times n$ linear equations

- 2 variables
- Solution is at the intersection of 2 lines



- 3 variables
- Solution is at the intersection of 3 planes



Matrix Form Representation of System of Linear Eqns

For the solution of a system of n equations (f_1, f_2, …f_n) in n state variables (x_1, x_2, … x_n)

$$\underline{f} = \begin{pmatrix} f_1(x_1,\, x_2,\, x_3\ldots,x_n) \\ f_2(x_1,\, x_2,\, x_3\ldots,x_n) \\ \vdots \\ f_n(x_1,\, x_2,\, x_3\ldots,x_n) \end{pmatrix}$$

The set of functions $\underline{f}(\underline{x_S},\underline{x_D})$ is to be solved such that $\underline{f}(\underline{x_S}) = \underline{0}$

Matrix Form Representation of System of Linear Eqns. …cont.

The system of equations $\underline{f}(\underline{x}) = \underline{0}$ can be expressed as:

$$\underline{f}(\underline{x}) = \underline{\underline{A}}\,\underline{x} - \underline{b} = \underline{0}$$

where,

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Matrix Form Representation of System of Linear Eqns. …cont.

$$\begin{aligned} x + y + z &= 4 \\ 2x + y + 3z &= 7 \\ 3x + y + 6z &= 5 \end{aligned} \equiv \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix} \equiv x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$

Matrix Rank

The rank of matrix A is defined as the order of the largest nonsingular square matrix within A. OR

The rank of a matrix is the maximum number of independent rows (or, the maximum number of independent columns).

When all of the vectors in a matrix are linearly independent, the matrix is said to be full rank.

Singular matrix is a matrix that has zero determinant.

Singular Rank

- A matrix that does not have an inverse is defined as singular.
- Properties of a singular matrix:
- A singular matrix has
  - A determinant of zero
  - Has rows that are linearly dependent
  - Has columns that are linearly dependent
  - Has no unique solution
  - The rank is less than n, number of rows

Singular Rank

Properties of singular matrices

- It has no inverse, A^{-1}
- Determinant is zero
- No unique solution to the system Ax=b
- Gaussian elimination can not avoid a zero on the diagonal
- Rank is less than n
- Rows and columns are linearly dependent

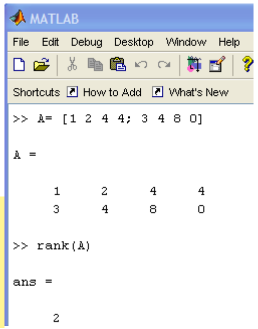
Properties of nonsingular matrices

- Its inverse, A^{-1}, exists
- Determinant is nonzero
- A unique solution to the system Ax=b exists
- Gaussian elimination does not encounter a zero on the diagonal
- Ranks equals n
- Rows and columns are not linearly dependent

Matrix Rank .. examples

$$X = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix}$$

```
>> rank([1 2 4 4; 3 4 8 0])
ans =
 2
```



Matrix Rank .. examples

$$Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 5 & 9 \end{bmatrix}$$

```
>> rank([1 2 3; 2 3 5; 3 4 7; 4 5 9])
ans =
   2
```

Since the matrix has more than zero elements, its rank must be greater than zero. And since it has fewer columns than rows, its maximum rank is equal to the maximum number of linearly independent columns.

Columns 1 and 2 are independent, because neither can be derived as a scalar multiple of the other. However, column 3 is linearly dependent on columns 1 and 2, because column 3 is equal to column 1 plus column 2. That leaves the matrix with a maximum of two linearly independent columns; e.g., column 1 and column 2. So the matrix rank is 2.

Matrix Rank .. examples

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Notice that row 2 of matrix A is a scalar multiple of row 1; that is, row 2 is equal to twice row 1. Therefore, rows 1 and 2 are linearly dependent. Matrix A has only one linearly independent row, so its rank is 1. Hence, matrix A is not full rank.

Now, look at matrix B. All of its rows are linearly independent, so the rank of matrix B is 3. Matrix B is full rank.

Matrix Rank .. determinant

The determinant is a unique number associated with a square matrix.

The rank r of matrix A is defined as the order of the largest nonsingular square matrix within A.

Singular matrix is a matrix that has zero determinant.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

The determinant is found by the formula: $|A| = (A_{11} * A_{22}) - (A_{12} * A_{21})$

Example:

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix} \qquad |A| = (5 * 6) - (1 * 2) = 30 - 2 = 28$$

```
>> det([5 1; 2 6])
ans =    28
```


Matrix Rank .. determinant

Example: consider the following (3×4) matrix:

$$\begin{bmatrix} 3 & 1 & 2 & -4 \\ 5 & 2 & 1 & 3 \\ 6 & 2 & 4 & -8 \end{bmatrix}$$

There are four sub-matrices of order (3×3):

$$\begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 1 \\ 6 & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & -4 \\ 5 & 2 & 3 \\ 6 & 2 & -8 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & -4 \\ 5 & 1 & 3 \\ 6 & 4 & -8 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 3 \\ 2 & 4 & -8 \end{bmatrix}$$

The determinant for each sub matrices:

12 0

Solution of a System of Equations: Matrix Inversion

The solution of the following system of equations

$$\underline{f}(\underline{x}) = \underline{\underline{A}}\,\underline{x} - \underline{b} = \underline{0}$$

can be written as follows:

$$\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

The objective is to find the inverse of Matrix A, i.e. A^{-1}

When a system of equations has at least one solution, it is said to be **consistent**; otherwise it is called **inconsistent**.

Inverse of a Matrix

$$A \cdot (adj\ A) = |A| \cdot I$$

Determinant of A

For a square matrix A

$$A^{-1} = \frac{1}{|A|}(adj\ A)$$

Adj is adjugate (or adjoint) of a square matrix which is a matrix that plays a role similar to the inverse of a matrix

The inverse of a square matrix A , sometimes called a reciprocal matrix, is a matrix such that $AA^{-1} = I$, where I is the identity matrix,

The identity matrix is a the simplest nontrivial diagonal matrix, $I(X) = X$

Linearity of a System of Equations

Although not explicitly specified, the solution method discussed in previous slide applies to "*linear*" system of equations.

What do we mean by "linear" systems?

Notes to be provided in class

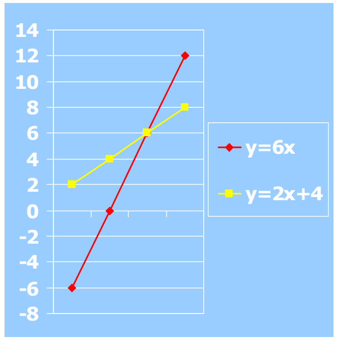
Linearity

A Linear Equation in 2 variables is of the form:

ax+by+c=0

Notice that this can be arranged into normal slope-intercept form:

y=mx+b



Linearity

An equation in n variables is said to be **linear** if it is equivalent to an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $x_1, x_2, \ldots, x_n$ are n distinct variables, $a_1, a_2, \ldots, a_n, b$ are constants, and at least one of the a's is not zero.

If each equation in a system of equations is linear, then we have a **system of linear equations**.

Independent Equations

Another implicit assumption made in deriving the solution for linear system of equations is that the equations are independent.

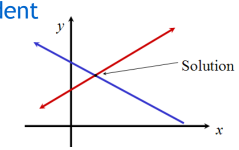
What does equations being independent mean?

How can a system of equation be independent of each other ?

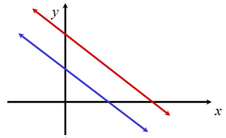
Notes to be provided in class

Consistent/inconsistent dependent/Independent

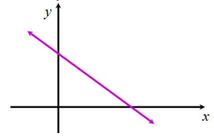
If the graph of the lines in a system of two linear equations in two variables intersect, then the system of equations has one solution, given by the point of intersection. The system is consistent and the equations are independent. Exactly one solution .



If the graph of the lines in a system of two linear equations in two variables are parallel, then the system of equations has no solution, because the lines never intersect. The system is inconsistent. No solution.



If the graph of the lines in a system of two linear equations in two variables are coincident, then the system of equations has infinitely many solutions, represented by the totality of points on the line. The system is consistent and dependent. Infinitely many solutions.



Consistency of equations and existence of solutions

The solution of a system of linear equations may or may not exist, and it may or may not be unique. Consider the following general system.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_M \end{bmatrix}$$

N **unknowns** are related by M **equations**

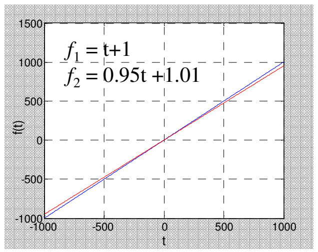
- ✓ M = N has unique solution if
  - no row is a linear combination of the others (row degeneracy), or
  - no column is a linear combination of the others (column degeneracy)
- ✓ Nonsingular matrix has a unique solution
- ✓ Mathematically these statements are exact,
- ✓ but, Numerically.....

Consistency of equations and existence of solutions

If a system is **too close to linear dependence** …

- an algorithm may **fail altogether** to get a solution
- **round off errors** can produce apparent linear dependence at some point in the solution process

⇒ The numerical procedure will fail totally



Consistency of equations and existence of solutions

If a system is **too close to linear dependence** …

- an algorithm may still work but **produce nonsense**
- **accumulated round of errors** can swamp the solution
  - particularly in close-to-singular systems
  - particularly if N is too large
- **not algorithmic failure**, but answer is (**wildly**) **incorrect**
- Error can be confirmed by direct substitution in original equations

Consistency of equations and existence of solutions

Under-determined systems…..

- M<N OR M = N with **degenerate equations**
- Fewer equations than unknowns
- May be **no solution**, OR
- May be an **infinite number** of solutions
  i.e. subspaces of solutions
  - arbitrary values must be assigned to (N-M) unknowns.
  - changing the values of (N-M) unknowns, results in new values for the rest of unknowns.
- **singular value decomposition** is a powerful technique

Consistency of equations and existence of solutions

Over-determined systems…..

- M>N AND **no degenerate**
- More equations than unknowns
- Generally there is **no solution**
- The **best compromise solution** is sought
  - closest to satisfy all equations
  - requires **quantification** of "closeness" to correct solution
  - most often: sum of squares of differences between left and right hand sides is minimized ⇒ **linear least squares problem**

Consistency of equations and existence of solutions

The solution of a system of linear equations may or may not exist, and it may or may not be unique. Existence of solutions can be determined by comparing the rank of the Matrix **A** with the rank of the **augmented matrix** $\mathbf{A_a}$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1N} \\ a_{21} & a_{22} & \dots a_{2N} \\ \dots\dots & \dots\dots & \dots\dots \\ a_{M1} & a_{M2} & \dots a_{MN} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_M \end{bmatrix} \quad \mathbf{A_a} = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1N} & b_1 \\ a_{21} & a_{22} & \dots a_{2N} & b_2 \\ \dots\dots & \dots\dots & \dots\dots & \dots \\ a_{M1} & a_{M2} & \dots a_{MN} & b_N \end{bmatrix}$$

N **unknowns** are related by M **equations**

Consistency of equations and existence of solutions

The set of equations has a solution if, and only if, the rank of the augmented matrix is equal to the rank of the coefficient matrix.

If in addition:

- ✓ rank (**A**) = N ⇒ **unique** solution
- ✓ rank (**A**) < N **&** rank (**A**) = rank ($\mathbf{A_a}$) ⇒ **Infinite number of** solutions
- ✓ rank (**A**) < N **&** rank (**A**) < rank ($\mathbf{A_a}$) ⇒ **No** solution

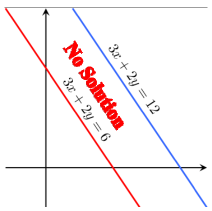
Example:

3x + 2y = 12

3x + 2y = 6

Rank(A)=1 < N, No solution

```
>> A=[3 2; 3 2]
>> b=[12; 6]
>> A\b
Warning: Matrix is singular to working precision.
ans =   Inf   -Inf
```



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- ✓ rank (**A**) < N **&** rank (**A**) < rank ($\mathbf{A_a}$) ⇒ **No** solution

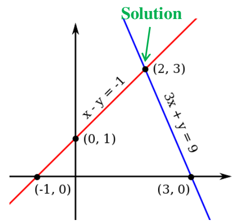
Example:

x - y = -1

3x + y = 9

Rank(A)=2 = N, unique solution

```
>> A=[1 -1; 3 1]
>> b=[-1; 9]
>> A\b
ans =
     2
     3
```



Methods for solving linear equations

$$a_1x_1 + a_2x_2 + \ \dots + a_nx_n = b$$

Direct methods:

It consists of techniques that do not require iterations to find the solution. Most direct methods involve manipulations of the matrix (**A**) and vector **b** to solve the equations.

Indirect (Iterative) methods:

These methods start with initial approximation (guess) $\mathbf{x^o}$ to the solution **x** and generate a sequence of vectors $\mathbf{x^k}$ that converge to the solution vector. These techniques are efficiently used to solve large systems where computer storage and computational time are important considerations.

Methods for solving linear equations

$$a_1x_1 + a_2x_2 + \ \dots + a_nx_n = b$$

There are several methods of solving systems of linear equations. Each is best used in different situations.

- Substitution Method
- Elimination Method
- Graphical method
- Matrix Algebra

Nowadays, easy access to computers makes the solution of very large sets of linear algebraic equations possible.

Substitution Method

Substitution method is used when it appears easy to solve for one variable in terms of the other. The goal is to reduce the system to two equations of one unknown each. Consider the following example:

$-2x + y = 4$

$-6x + y = 0$

$y = 6x$

$-2x + 6x = 4$

$4x = 4$

$x = 1$ then $y = 6(x) = 6$

Elimination Method

Elimination method is used when it appears easy to eliminate one variable from the system through transformation. Remember that linear transformations do not change the solutions of a system.

Examine the original system:

$-2x + y = 4$

$-6x + y = 0$

Let's transform the second equation:

$-1(-6x + y = 0)$

$6x - y = 0$

Now add the equations together:

$$\begin{array}{rcrcl} & -2x & + y & = & 4 \\ + & 6x & - y & = & 0 \\ \hline & 4x & & = & 4 \end{array}$$

$x = 1$ then $y = 6(x) = 6$

Matrix Algebra

Linear algebra is used to solve a system of linear equations. First, take the x and y coefficients and place them in a matrix:

$$\begin{array}{l} -2x + y = 4 \\ -6x + y = 0 \end{array} \longrightarrow \begin{bmatrix} -2 & 1 \\ -6 & 1 \end{bmatrix}$$

Next, place the constant terms into a vector:

$$\begin{array}{l} -2x + y = 4 \\ -6x + y = 0 \end{array} \longrightarrow \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Then augment the matrix and vector:

$$\left[\begin{array}{cc|c} -2 & 1 & 4 \\ -6 & 1 & 0 \end{array}\right]$$

Example …. Elimination Method

$$\text{Solve:} \begin{cases} x + y + z = 4 & (1) \\ 2x - y - z = 2 & (2) \\ -x + 2y + 3z = 5 & (3) \end{cases}$$

Step 1: Multiply (1) by -2 get $-2x - 2y - 2z = -8$ and add to (2).

$$\begin{cases} x + y + z = 4 & (1) \\ \quad\; - 3y - 3z = -6 & (2) \\ -x + 2y + 3z = 5 & (3) \end{cases}$$

Example …. Elimination Method … cont.

Step 2: Add (1) to (3).

$$\begin{cases} x + y + z = 4 & (1) \\ \quad -3y - 3z = -6 & (2) \\ \quad\quad 3y + 4z = 9 & (3) \end{cases}$$

Step 3: Add (2) to (3).

$$\begin{cases} x + y + z = 4 & (1) \\ \quad -3y - 3z = -6 & (2) \\ \quad\quad\quad z = 3 & (3) \end{cases}$$

Example …. Elimination Method … cont.

Step 4: Now that z=3 substitute that into (2) to solve for y.

$$-3y - 3*3 = -6$$
$$-3y - 9 = -6$$
$$-3y = 3$$
$$y = -1$$

Step 5: Substitute z=3 and y=-1 into (1) and solve for x.

$$x - 1 + 3 = 4$$
$$x + 2 = 4$$
$$x = 2$$

$\boxed{\text{Solution } (x, y, z) = (2, -1, 3).}$

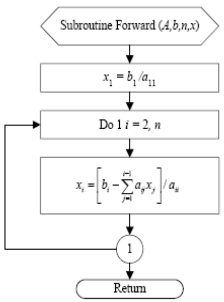
Elimination Method - algorithm

Forward Substitution

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j\right) \qquad i = 2, 3, \cdots, n$$



Forward Subsitution Subroutine

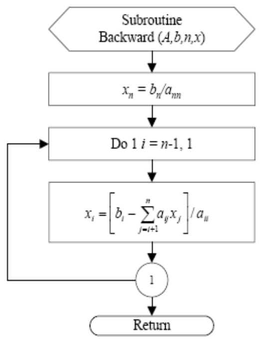
Elimination Method - algorithm

Backward Substitution

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j\right) \qquad i = n-1, n-2, \cdots, 1$$



Backward Subsitution Subroutine

Gauss Elimination

- It systematically eliminates unknowns from the problem till only one equation and unknown are left.
- The value of this unknown is determined and the remaining unknowns are calculated in turn.
- Gaussian elimination is very efficient method in solving large sets of linear equations. For large n, it requires $n^3/2$ calculations to get the solution.

Gauss Elimination

- Aim: try to convert an augmented form matrix to an upper triangular form by using elementary row operations.
- Elementary row operations
  - multiply any row of the augmented coefficient matrix by a constant.
  - add a multiple of one row to a multiple of any other row.
  - may interchange the order of any two rows.

Gaussian elimination…

- Consider a system of 3×3 linear equations in matrix form, **Ax** = **b**:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- To make book-keeping simpler, we represent the system by an augmented matrix:

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

…Gaussian elimination…

- The first column can be made zero by subtracting a_{21}/a_{11} times the first row from the second row, and subtracting a_{31}/a_{11} times the first row from the third row (primes indicate changed values)

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

- Similarly, the second column can be made zero by subtracting a'_{32}/a'_{22} times the first row from the third row (double primes indicate changed values), forming an upper triangular matrix:

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

…Gaussian elimination…

- The last row represents an equation in a single variable

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

  $a''_{33}\, x_3 = b''_3$
  which can be solved as $x_3 = b''_3 \,/\, a''_{33}$
- The second row represents an equation in two variables
  $a'_{22}\, x_2 + a'_{23}\, x_3 = b'_2$
- Since the variable x_3 has already been found in the previous step, x_2 can be solved as
  $x_2 = (b'_2 - a'_{23}\, x_3) \,/\, a'_{22}$

…Gaussian elimination

- The first row represents an equation in three variables
  $a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 = b_1$
- Since the variables x_2 and x_3 have already been found in the previous steps, x_1 can be solved as
  $x_1 = (b_1 - a_{12}\, x_2 - a_{13}\, x_3) \,/\, a_{11}$
- This process of solving an upper triangular matrix equation is called **back substitution**.

Gauss Elimination

- Solve $Ax = b$
- Consists of two phases:
  - **–Forward elimination**
  - **–Back substitution**
- *Forward Elimination* reduces $Ax = b$ to an upper triangular system $Tx = b'$
- *Back substitution* can then solve $Tx = b'$ for x

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \Downarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Forward Elimination

$\Downarrow$

$$x_3 = \frac{b''_3}{a''_{33}} \qquad x_2 = \frac{b'_2 - a'_{23}x_3}{a'_{22}}$$

$$x_1 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$$

Back Substitution

Gauss Elimination

Pivot element

Pivot row

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Use row operations to simplify the system. e.g. eliminate element A_{21} by subtracting $A_{21}/A_{11} = d_{21}$ times row 1 from row 2.

There is no restriction upon which equation is placed first or which variable is eliminated first….

Forward Elimination

$$\begin{array}{r} -(3/1) \\ -(2/1) \end{array} \quad \begin{aligned} x_1 - x_2 + x_3 &= 6 \\ 3x_1 + 4x_2 + 2x_3 &= 9 \\ 2x_1 + x_2 + x_3 &= 7 \end{aligned} \quad \Rightarrow \quad -(3/7) \quad \begin{aligned} x_1 - x_2 + x_3 &= 6 \\ 0 + 7x_2 - x_3 &= -9 \\ 0 + 3x_2 - x_3 &= -5 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 - x_2 + x_3 &= 6 \\ 0 \quad 7x_2 - x_3 &= -9 \\ 0 \quad 0 \quad -(4/7)x_3 &= -(8/7) \end{aligned}$$

Solve using BACK SUBSTITUTION: $x_3 = 2$ $x_2 = -1$ $x_1 = 3$

$$a_{ji} = a_{ji} + a_{ii}\left(\frac{-a_{ji}}{a_{ii}}\right) = 0$$

Total No. of operations for forward Elimination:

$$2n^2 + 2(n-1)^2 + \ldots + 2*(2)^2 + 2*(1)^2 = 2\sum_{i=1}^{n} i^2 = 2\frac{n(n+1)(2n+1)}{6}$$

…Gaussian elimination: example 1

- Back substitute

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 0 & -7 & -7 \end{array}\right]$$

- The last row represents an equation in a single variable, $a''_{33}\, z = b''_3$, which can be solved as $z = b''_3 / a''_{33} = -7 / -7 = 1$
- The second row represents an equation in two variables, $a'_{22}\, y + a'_{23}\, z = b'_2$, which can be solved as $y = (b'_2 - a'_{23}\, z) / a'_{22} = (-6 + 2) / (-2) = 2$
- The first row represents an equation in three variables $a_{11}\, x + a_{12}\, y + a_{13}\, z = b_1$, which can be solved as $x_1 = (b_1 - a_{12}\, y - a_{13}\, z) / a_{11} = (4 - 2 - 1) / 1 = 1$
- The solution is thus $x = 1$, $y = 2$, $z = 1$.

Gaussian elimination: example 2…

- Solve the system of equations

$$\begin{aligned} 2x + 5y + 3z &= 7 \\ x - 2.5y - 1.5z &= -7.5 \\ 2x + 7y + 4.5z &= 12 \end{aligned}$$

- Represent the system as an augmented matrix:

$$\left[\begin{array}{ccc:c} 2 & 5 & 3 & 7 \\ 1 & -2.5 & -1.5 & -7.5 \\ 2 & 7 & 4.5 & 12 \end{array}\right]$$

…Gaussian elimination: example 2…

- Calculate the row multipliers and record them in the **L** matrix, $l_{21} = a_{21}/a_{11}$ and $l_{31} = a_{31}/a_{11}$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & \bullet & 1 \end{bmatrix}$$

- Zero the first column by subtracting $l_{21} = a_{21}/a_{11}$ times the first row from the second row, and subtracting $l_{31} = a_{31}/a_{11}$ times the first row from the third row

$$\left[\begin{array}{ccc:c} 2 & 5 & 3 & 7 \\ 1 & -2.5 & -1.5 & -7.5 \\ 2 & 7 & 4.5 & 12 \end{array}\right] \to \left[\begin{array}{ccc:c} 2 & 5 & 3 & 7 \\ 0 & -5 & -3 & -11 \\ 0 & 2 & 1.5 & 5 \end{array}\right]$$

…Gaussian elimination: example 2…

- Calculate the row multiplier and record it in the **L** matrix, $l_{32} = a'_{32}/a'_{22}$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & -0.4 & 1 \end{bmatrix}$$

- Zero the second column by subtracting $l_{32} = a'_{32}/a'_{22}$ times the second row from the third row, forming an upper triangular matrix

$$\left[\begin{array}{ccc:c} 2 & 5 & 3 & 7 \\ 0 & -5 & -3 & -11 \\ 0 & 2 & 1.5 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 2 & 5 & 3 & 7 \\ 0 & -5 & -3 & -11 \\ 0 & 0 & 0.3 & 0.6 \end{array}\right]$$

…Gaussian elimination: example 2

- Back substitute

$$\left[\begin{array}{ccc:c} 2 & 5 & 3 & 7 \\ 0 & -5 & -3 & -11 \\ 0 & 0 & 0.3 & 0.6 \end{array}\right]$$

- The last row can be solved as $z = b''_3 \,/\, a''_{33} = 0.6 \,/\, 0.3 = 2$
- The second row can be solved as $y = (b'_2 - a'_{23}\, z) \,/\, a'_{22} = (-11 + 6) \,/\, (-5) = 1$
- The first row can be solved as $x = (b_1 - a_{12}\, y - a_{13}\, z) \,/\, a_{11}$
  $= (7 - 5 - 6) \,/\, 2 = -2$
- The solution is thus $x = -2$, $y = 1$, $z = 2$.

Gaussian elimination: example 3…

- Solve the system of equations

$$\begin{aligned} 0.0001x + 0.0001y + 1.99z &= 10 \\ 2x + 2.001y + z &= 1 \\ 4x + 3y + 2.982z &= 1 \end{aligned}$$

- Work to 4 significant figures and give the answer to 3. Check the answer by substituting into the original equation.
- Represent the system as an augmented matrix:

$$\left[\begin{array}{ccc:c} 0.0001 & 0.0001 & 1.99 & 10 \\ 2 & 2.001 & 1 & 1 \\ 4 & 3 & 9.282 & 1 \end{array}\right]$$

…Gaussian elimination: example 3…

- Calculate the row multipliers and record them in the **L** matrix, $l_{21} = a_{21}/a_{11}$ and $l_{31} = a_{31}/a_{11}$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 20{,}000 & 1 & 0 \\ 40{,}000 & \bullet & 1 \end{bmatrix}$$

- Zero the first column by subtracting l_{21} times the first row from the second row, and subtracting l_{31} times the first row from the third

$$\left[\begin{array}{ccc:c} 0.0001 & 0.0001 & 1.99 & 10 \\ 2 & 2.001 & 1 & 1 \\ 4 & 3 & 9.282 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc:c} 0.0001 & 0.0001 & 1.99 & 10 \\ 0 & 0.001 & -39{,}800 & -200{,}000 \\ 0 & -1 & -79{,}600 & -400{,}000 \end{array}\right]$$

…Gaussian elimination: example 3…

- Calculate the row multiplier and record it in the **L** matrix, $l_{32} = a'_{32}/a'_{22}$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 20,000 & 1 & 0 \\ 40,000 & -1,000 & 1 \end{bmatrix}$$

- Zero the second column by subtracting $l_{32} = a'_{32}/a'_{22}$ times the second row from the third row, forming an upper triangular matrix

$$\left[\begin{array}{ccc|c} 0.0001 & 0.0001 & 1.99 & 10 \\ 0 & 0.001 & -39,800 & -200,000 \\ 0 & -1 & -79,600 & -400,000 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 0.0001 & 0.0001 & 1.99 & 10 \\ 0 & 0.001 & -39,800 & -200,000 \\ 0 & 0 & -39,880,000 & -200,400,000 \end{array}\right]$$

…Gaussian elimination: example 3…

- Back substitute

$$\left[\begin{array}{ccc|c} 0.0001 & 0.0001 & 1.99 & 10 \\ 0 & 0.001 & -39,800 & -200,000 \\ 0 & 0 & -39,880,000 & -200,400,000 \end{array}\right]$$

- The last row can be solved as $z = -39,880,000 / -200,400,000 = 5.025$
- The second row can be solved as $y = (-200,000 + 199,995) / (0.001) = -5,000$
- The first row can be solved as $x = (b_1 - a_{12}\, y - a_{13}\, z) / a_{11}$ $= (10 + 0.5 - 9.99975) / 0.0001 = 5,003$
- The computed solution is thus $x = 5,003$, $y = -5,000$, $z = 5.03$

…Gaussian elimination: example 4

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{bmatrix} \Rightarrow \begin{matrix} \\ 3R_1 + 4R_2 \\ R_1 - 4R_3 \end{matrix} \begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 2 & -11 & -37 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} \\ \\ R_2 + 5R_3 \end{matrix} \begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & -36 & -108 \end{bmatrix}$$

Solution:

- $x_3=3$
- $x_2=-2$
- $x_1=2$

Gauss Elimination - problems

Division by zero

It is possible that during both elimination and back-substitution phases a division by zero can occur.

For example:

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 4x_1 + 6x_2 + 7x_3 &= -3 \\ 2x_1 + x_2 + 6x_3 &= 5 \end{aligned} \qquad A = \begin{matrix} 0 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 1 & 6 \end{matrix}$$

Gauss Elimination - problems

Division by zero

- During the triangularization step, if a zero is encountered on the diagonal, that row can not be used to eliminate coefficients below that zero element.
- However, in that case, we can continue by interchanging rows and eventually achieve an upper triangular matrix of coefficients.
- A useful strategy to avoid zero divisors is to rearrange the equations so as to put the coefficient of largest magnitude on the diagonal at each step. This is called "pivoting".
- The real stumbling block is finding a zero on the diagonal after we have triangularized.
- If that occurs, there is no solution and determinant is zero.

Gauss Elimination - problems

Pivoting

- A useful strategy to avoid zero divisors is to rearrange the equations so as to put the coefficient of largest magnitude on the diagonal at each step. This is called "pivoting".
- Complete pivoting may require both row and column interchanges.
- Partial pivoting, which places a coefficient of larger magnitude on the diagonal by row interchanges only, will guarantee a nonzero divisor if there is a solution to the sets of equation and will have the added advantage of giving improved arithmetic precision.

Gauss Elimination - problems

ill-conditioned systems - small changes in coefficients result in large changes in the solution. Alternatively, a wide range of answers can approximately satisfy the equations.

(*Well-conditioned systems* – small changes in coefficients result in small changes in the solution)

Problem: Since *round off errors* can induce small changes in the coefficients, these changes can lead to large solution errors in *ill-conditioned* systems.

Example:

$$x_1 + 2x_2 = 10$$
$$\mathbf{1.1}x_1 + 2x_2 = 10.4$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{\begin{vmatrix} 10 & 2 \\ 10.4 & 2 \end{vmatrix}}{1(2) - 2(1.1)} = \frac{2(10) - 2(10.4)}{-0.2} = 4 \qquad x_2 = 3$$

$$x_1 + 2x_2 = 10$$
$$\mathbf{1.05}x_1 + 2x_2 = 10.4$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{\begin{vmatrix} 10 & 2 \\ 10.4 & 2 \end{vmatrix}}{1(2) - 2(1.05)} = \frac{2(10) - 2(10.4)}{-0.1} = 8 \qquad x_2 = 1$$

Gauss Elimination - problems

Wrong solution

$$\begin{array}{l} 0.01x_1 - x_2 = 1 \\ x_1 + 0.01x_2 = 1 \end{array} \Rightarrow \begin{array}{ccc} 0.01 & -1.0 & 1.0 \\ 1.0 & 0.01 & 1.0 \end{array} \Rightarrow \begin{array}{ccc} 1.0 & -100 & 100 \\ 1.0 & 0.01 & 1.0 \end{array} \Rightarrow \begin{array}{ccc} 0.01 & -1.0 & 1.0 \\ 0.0 & -100.01 & 99 \end{array}$$

$x_2 = 99/-100.01 =$ **- 0.9899** and $x_1 = (1-1)/0.01 =$ **0.0**

Re-arrange the equations

$$\begin{array}{l} x_1 + 0.01x_2 = 1 \\ 0.01x_1 - x_2 = 1 \end{array} \Rightarrow \begin{array}{ccc} 1.0 & 0.01 & 1.0 \\ 0.01 & -1.0 & 1.0 \end{array} \Rightarrow \begin{array}{ccc} 0.01 & 0.0001 & 0.01 \\ 0.01 & -1.0 & 1.0 \end{array} \Rightarrow \begin{array}{ccc} 0.01 & 0.0001 & 0.01 \\ 0.0 & 1.0001 & -0.99 \end{array}$$

$x_2 = -0.99/1.0001 =$ **-0. 9899** and $x_1 = (0.01+0.0001x0.9899)/0.01 =$ **1.00989**

Exact solution…

$x_2 =$ **-0.9899** and $x_1 =$ **1.0099**

LU Factorization
LU factorization (without pivoting)

Let:

A = L.U

with **L** unit lower triangular, **U** upper triangular, i.e.

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

LU Factorization
LU factorization (without pivoting)

This factorization can be used to solve the following linear set

A.x = (L.U).x = L.(U.x) = L.y = b

Such that

L.y = b (forward substitution)
U.x = y (backward substitution)

How many set of linear equations do we have now?

What is the advantage of breaking up one linear set into two successive ones?

LU Factorization
Algorithm….

This method is called Doolittle's factorization

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Partition **A**, **L**, **U** as block matrices:

$$A=\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad L=\begin{bmatrix} 1 & 0 \\ L_{21} & L_{22} \end{bmatrix}, \qquad U=\begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

with $\mathbf{L_{22}}$ unit-lower triangle and $\mathbf{U_{22}}$ upper triangle

LU Factorization

Determine **L**, **U** from **A = L.U**, i.e.

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11} & U_{12} \\ u_{11}L_{21} & L_{21}U_{12}+L_{22}U_{22} \end{bmatrix}$$

this gives

$u_{11}= a_{11}$, $\quad \mathbf{U_{12} = A_{12}}$, $\quad \mathbf{L_{21}} = (1/a_{11})\mathbf{A_{21}}$

$\mathbf{L_{22}U_{22} = A_{22} - L_{21}U_{12}}$

LU Factorization

Example….

Factorize matrix **A**

$$A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}$$

Solution…

$$A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

LU Factorization

Solution…

Step 1: First row of **U** and first column of **L**

$$\begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 & 9 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Step 2: Find l_{32}, u_{22}, u_{23} and u_{33}

$$\mathbf{L_{22}U_{22} = A_{22} - L_{21}U_{12}}$$

$$\begin{bmatrix} 1 & 0 \\ l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{22} & u_{23} \\ 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \begin{bmatrix} 2 & 9 \end{bmatrix}$$

LU Factorization

Solution…

Step 2: Find l_{32}, u_{22}, u_{23} and u_{33}

$$\begin{bmatrix} 1 & 0 \\ l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{22} & u_{23} \\ 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \begin{bmatrix} 2 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 11/16 & 1 \end{bmatrix} \begin{bmatrix} 8 & -1/2 \\ 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 8 & -1/2 \\ 11/2 & 9/4 \end{bmatrix}$$

$$u_{33} = 9/4 + 11/32 = 83/32$$

$$A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & 11/16 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 & 9 \\ 0 & 8 & -1/2 \\ 0 & 0 & 83/32 \end{bmatrix}$$

LU Factorization

Note….

- ❑ Re-solve the previous example using GE method

$$A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}$$

What are the values for d_{ij}?

$d_{21} = 1/2$

$d_{31} = 6/8$

$d_{32} = 11/16$

Compare

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & 11/16 & 1 \end{bmatrix}$$

LU Factorization

Note….

- Re-solve the previous example using GE method

$$A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}$$

After elimination steps we got

$$\begin{bmatrix} 8 & 2 & 9 \\ 0 & 8 & -1/2 \\ 0 & 0 & 83/32 \end{bmatrix} \quad \text{Compare} \quad \begin{bmatrix} 8 & 2 & 9 \\ 0 & 8 & -1/2 \\ 0 & 0 & 83/32 \end{bmatrix}$$

LU Factorization

Note….

- Solving a set of equations using LU decomposition takes the same number of calculations as Gaussian elimination since both methods are essentially equivalent. Why to use this method?

1. The solution of a triangular set of equations is quite trivial, forward substitution is used to find **y** elements and vector **x** is obtained using back substitution scheme

$$x_n = y_n / u_{nn}$$

$$x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j\right) \qquad i = n-1, n-2, \ldots, 1$$

$$y_1 = b_1 / l_{11}$$

$$y_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij} y_j\right) \qquad i = 1, 2, 3, \ldots, n$$

LU Factorization

Note….

- Solving a set of equations using LU decomposition takes the same number of calculations as Gaussian elimination since both methods are essentially equivalent. Why to use this method?

2. Once **A** is transformed into **L** and **U**, it can be used with any right-hand-side vector, **b**.

LU Factorization

MATLAB has built in routines for solving linear systems. For LU decomposition, use the command “lu”. Consider the previous example

$$A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}$$

```
A =[8 2 9;4 9 4;6 7 9];
[L,U]=lu(A)
b=[1;2;3];
Y=L\b;
X=U\Y
```

LU Factorization

Example 2.4 Linear Equations from Material Balances

Problem Statement

A process for producing pasteboard from waste wood by removing some water and adding glue is shown in **Figure 2.3**. The process operates such that ⅔ of the water entering the evaporator is removed in stream V. Also, ½ pound of glue is added in Mixer 1 for every pound of water that enters Mixer 1 (in stream C). Calculate all unknown flow rates.

First, let us identify the unknowns of the problem: A, B, V, C_W, C_S, C_{WD}, E, G_W, G_S, G_{WD}, G_G, P_{WD}, P_W, P_S and P_G, where the subscripted symbols (W, WD, S, G) indicate the amount of water, wood, salt, and glue, respectively, in a particular stream. Therefore, there are 15 unknowns in this problem.

Now let us count the number of independent equations. You can write an independent mass balance equation for each component in each unit except for the divider. Therefore, you can write only one independent equation for a divider, three for the evaporator, and four each for Mixers 1 and 2, for a total of 12 independent equations. In addition, the composition of the glue in stream P (see **Figure 2.3**) provides the following relationship:

$$\frac{P_G}{P_{WD}+P_W+P_S+P_G} = 0.03$$

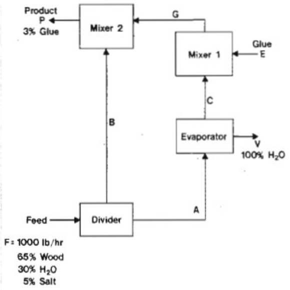


Figure 2.3 Schematic of a pasteboard process.

LU Factorization

Example 2.4

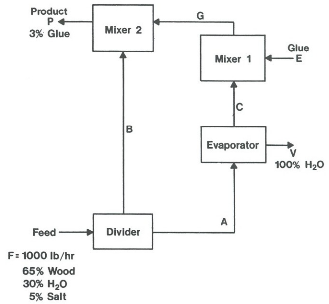


Figure 2.3 Schematic of a pasteboard process.

$A + B = 1000$	(Divider - overall balance)
$A = V + C_W + C_S + C_{WD}$	(Evaporator - overall balance)
$0.65A = C_{WD}$	(Evaporator - wood balance)
$0.05A = C_S$	(Evaporator - salt balance)
$C_W = G_W$	(Mixer 1 - water balance)
$C_S = G_S$	(Mixer 1 - salt balance)
$C_{WD} = G_{WD}$	(Mixer 1 - wood balance)
$E = G_G$	(Mixer 1 - glue balance)
$0.3B + G_W = P_W$	(Mixer 2 - water balance)
$0.65B + G_{WD} = P_{WD}$	(Mixer 2 - wood balance)
$0.05B + G_S = P_S$	(Mixer 2 - salt balance)
$G_G = P_G$	(Mixer 2 - glue balance)

$$V = \frac{2}{3}(0.3A)$$

$$E = 0.5C_W$$

LU Factorization

Example 2.4

x Unknowns

A	B	V	C_W	C_S	C_{WD}	E	G_W	G_S	G_{WD}	G_G	P_{WD}	P_W	P_S	P_G	**b**
A	$+B$														$= 1000$
A		$-V$	$-C_W$	$-C_S$	$-C_{WD}$										$= 0$
$0.65A$					$-C_{WD}$										$= 0$
$0.05A$				$-C_S$											$= 0$
			C_W				$-GW$								$= 0$
				C_S				$-GS$							$= 0$
					C_{WD}				$-GWD$						$= 0$
						E				$-G_G$					$= 0$
	$0.3B$						$+G_W$					$-PW$			$= 0$
	$0.65B$								$+G_{WD}$		$-PWD$				$= 0$
	$0.05B$							$+G_S$					$-P_S$		$= 0$
										G_G				$-P_G$	$= 0$
											$-0.03P_{WD}$	$-0.03P_W$	$-0.03P_S$	$+0.97P_G$	$= 0$
$-⅔(0.3A)$		$+V$													$= 0$
			$-0.5C_W$			E									$= 0$

A

LU Factorization

Example 2.4

PROGRAM LISTING FOR EXAMPLE 2.4

```
C
C***********************  ABSTRACT  ***********************
C
C      THIS PROGRAM SOLVES THE MATERIAL BALANCE PROBLEM
C   POSED IN EXAMPLE 2.4 USING THE LINEAR EQUATION
C   SOLVER LINPAC.
C
C*********************  NOMENCLATURE  *********************
C
C   A(I,J)- THE COEFFICIENT OF THE JTH VARIABLE IN THE
C           ITH EQUATION
C   B(I)- THE CONSTANT TERM IN THE ITH EQUATION
C   N- THE NUMBER OF INDEPENDENT EQUATIONS
C   X(I)- THE VECTOR CONTAINING THE SOLUTION TO THE SET
C         OF LINEAR EQUATIONS
C   IPVT(I)- WORK VECTOR
C
C**********************************************************
C
      DOUBLE PRECISION A(15,15),B(15),X(15 ,IPVT(15)
C   INPUT DATA FOR THE PROBLEM
      N=15
C   ZERO B(I) AND A(I,J)
      DO 1 I=1,N
      B(I)=0.0
      DO 1 J=1,N
    1 A(I,J)=0.0
C   SET THE NONZERO VALUES OF A(I,J) AND B(I)
```

```
C
C   CALL LINPAC
C
      CALL LINPAC(N,A,B,X,IPVT)
C
C   PRINT OUT RESULTS
C
      WRITE(6,88)
   88 FORMAT( ///)
      DO 20 I=1,N
   20 WRITE(6,10)I,X(I)
      WRITE(6,88)
   10 FORMAT( 10X,3H X(,I2,3H)= ,D14.7)
      STOP
      END
```

LINPAC (N, A, B, X, IPVT)

USE

Solves a set of linear equations.

ARGUMENTS

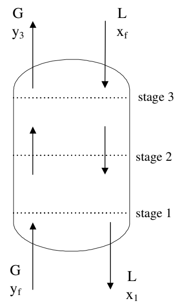
N	The number of equations (input).
A(I,J)	The coefficient of the jth variable in the ith equation. This matrix *must be* double precisioned and dimensioned N × N (input).
B(I)	The constant term for the ith equation. This vector *must be* double precisioned and dimensioned by N (input).
X(I)	The value of the ith unknown. This vector *must be* double precisioned and dimensioned by N (output).
IPVT(I)	A work vector which must be double precisioned and dimensioned by N.

LU Factorization

Example …

A three-stages absorption system, shown in the figure below, is described by the following set of equations:

$$(L+aG)x_1 - aGx_2 = Lx_f$$
$$Lx_1 - (aG+L)x_2 + aGx_3 = 0$$
$$Lx_2 - (L+aG)x_3 = -Gy_f$$



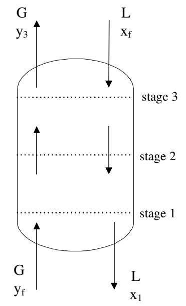
Solve for the values of x_i and y_i, for the following data: $x_f = 0.01$, $y_f = 0.06$, L = 40.8 lb/min, G = 66.7 lb/min, a=0.72. Note that $y_i = ax_i$

LU Factorization

Solution

$$(40.8+0.72\cdot 66.7)x_1 - 0.72\cdot 66.7x_2 = 40.8\cdot 0.01$$
$$40.8\cdot x_1 - (0.72\cdot 66.7+40.8)x_2 + 0.72\cdot 66.7x_3 = 0$$
$$40.8\cdot x_2 - (0.72\cdot 66.7+40.8)x_3 = -66.7\cdot 0.06$$

$$88.82\cdot x_1 - 48.02.7x_2 = 0.41$$
$$40.80\cdot x_1 - 88.82\cdot x_2 + 48.02.7x_3 = 0$$
$$40.80\cdot x_2 - 88.82\cdot x_3 = -4.00$$



Use MATLAB…. Write your code

LU Factorization

System Condition

$$x_1 + x_2 = 2$$
$$x_1 + 1.0001x_2 = 2.0001$$
Solution : $x_1 = 1$ and $x_2 = 1$

$$x_1 + x_2 = 2$$
$$x_1 + 0.9999x_2 = 2.0001$$
Solution : $x_1 = 3$ and $x_2 = -1$

$$x_1 + x_2 = 2$$
$$x_1 + 1.0001x_2 = 2$$
Solution : $x_1 = 2$ and $x_2 = 0$

Determinant = 1×10^{-4}

Condition number ~ 10^4,It is *ill-conditioned* system

LU Factorization

System Condition

- A system whose coefficient matrix is nearly singular is ill-conditioned.
- When a system is ill conditioned, the solution is very sensitive to changes in the right-hand side vector.
- It is also sensitive to small changes in the coefficients.
- For an ill-conditioned system, small changes in the input make large changes in the output.

LU Factorization

System Condition – another example

- 1.01x + 0.99y = b1
- 0.99x + 1.01y = b2
- solution : **x=1, y=1** for b is [2.00 2.00]
- If b is [2.02 1.98] solution becomes **x=2, y=0**.
- If b is [1.98 2.02] it would be **x=0, y=2**.
- It must be apparent that while solving an ill-conditioned system, the round-off or truncation errors yield very different results.

Gauss-Seidel Method
An iterative approach

Why to use this method?

- It requires fewer calculations than GE for very large systems of linear equations or for linear equations with a sparse coefficient matrix which is not banded.
- It is an efficient method for the solution of ill-conditioned set of linear equations.
- If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss-Seidel Method

Basic Procedure

- Algebraically solve each linear equation for x_i.
- Assume an initial guess solution array.
- Solve for each x_i.
- Check for error after each iteration to check if error is within a pre-specified tolerance.
- Repeat until error is within the tolerance.

Gauss-Seidel Method

Algorithm

A set of n equations and n unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

If the diagonal elements are non-zero
Rewrite each equation solving for the corresponding unknown

For example:
- First equation, solve for x_1
- n equation, solve for x_n

Gauss-Seidel Method

Algorithm

Rewriting each equation

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}} \longleftarrow \text{From Equation 1}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}} \longleftarrow \text{From equation 2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}} \longleftarrow \text{From equation n-1}$$

$$x_n = \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}} \longleftarrow \text{From equation n}$$

Gauss-Seidel Method

Algorithm

General Form for any row 'i'

$$x_i = \frac{b_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Gauss-Seidel Method

Algorithm

Solve for the unknowns

Assume an initial guess for [X]

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Use rewritten equations to solve for each value of x_i.

Remember to use the <u>*most recent*</u> *value of x_i.*

Gauss-Seidel Method

Algorithm

Check for the tolerance using a pre-defined criterion such as

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

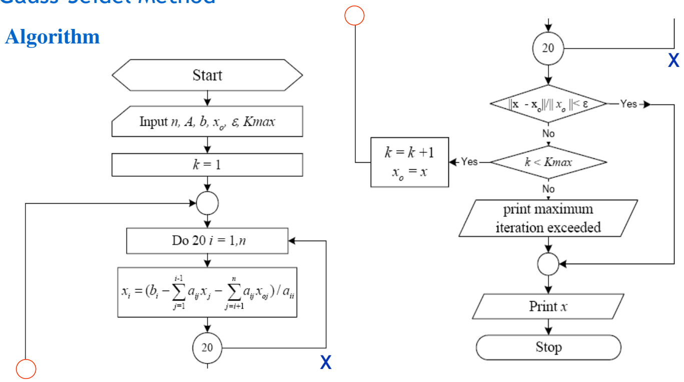
$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{old}}\right| \times 100$$

$$\left|\in_a\right|_i = \left|x_i^{new} - x_i^{old}\right|$$

The iterations are stopped when the error is less than a pre-specified tolerance for <u>all</u> unknowns.

Gauss-Seidel Method

Algorithm



$$x_i = (b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_{oj}) / a_{ii}$$

Gauss-Seidel Method

Example….

Consider the following set of linear equations:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$x_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$x_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$x_3 = 279.2 - 144a_1 - 12a_2$$

Gauss-Seidel Method

Example…. Solution … cont.

Applying the initial guess and solving for x_i

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_o = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Initial Guess

$$x_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$x_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510$$

$$x_3 = 279.2 - 144(3.6720) - 12(-7.8510) = -155.36$$

When solving for x_2, how many of the initial guess values were used?

Gauss-Seidel Method

Example…. Solution … cont.

Finding the percentage relative error

$$\left|\epsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\epsilon_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| x100 = 72.76\%$$

$$\left|\epsilon_a\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| x100 = 125.47\%$$

$$\left|\epsilon_a\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| x100 = 103.22\%$$

At the end of the first iteration

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

The maximum percentage relative error is 125.47%

Gauss-Seidel Method

Example.... Solution ... cont.

Iteration #2

Using

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

from iteration #1

the values of a_i are found:

$$x_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$

$$x_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$

$$x_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

Gauss-Seidel Method

Example.... Solution ... cont.

Finding the percentage relative error

$$|\epsilon_a|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| \cdot 100 = 69.543\%$$

$$|\epsilon_a|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| \cdot 100 = 85.695\%$$

$$|\epsilon_a|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| \cdot 100 = 80.540\%$$

At the end of the second iteration

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

The maximum percentage relative error is 85.695%

Gauss-Seidel Method

Example.... Solution ... cont.

Repeating more iterations, we got

Iteration	x_1	$\lvert\epsilon_a\rvert_1 \%$	x_2	$\lvert\epsilon_a\rvert_2 \%$	x_3	$\lvert\epsilon_a\rvert_3 \%$
1	3.6720	72.767	−7.8510	125.47	−155.36	103.22
2	12.056	69.543	−54.882	85.695	−798.34	80.540
3	47.182	74.447	−255.51	78.521	−3448.9	76.852
4	193.33	75.595	−1093.4	76.632	−14440	76.116
5	800.53	75.850	−4577.2	76.112	−60072	75.963
6	3322.6	75.906	−19049	75.972	−249580	75.931

Notice – The relative errors are not decreasing

Also, the solution is not converging to the true solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.690 \\ 1.0857 \end{bmatrix}$$

Gauss-Seidel Method

What is the problem?

This example illustrates a drawback of the Gauss-Siedel method: *not all systems of equations will converge.*

Is it fixable?

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: **A** in **A.X** = **b** is diagonally dominant if:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad i = 1, 2, \ldots, n$$

Gauss-Seidel Method

Which coefficient matrix is diagonally dominant?

$$[A]=\begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix} \qquad [B]=\begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.

Gauss-Seidel Method

Example 2

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_o = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The coefficient matrix is:

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

Will the solution converge using the Gauss-Siedel method?

Gauss-Seidel Method

Example 2 - solution

Checking if the coefficient matrix is diagonally dominant

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$
$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$
$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true; therefore the solution should converge using the Gauss-Seidel Method

Gauss-Seidel Method

Example 2 – solution … cont.

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_o = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Gauss-Seidel Method

Example 2 – solution … cont.

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000-1.0000}{0.50000}\right|\times 100 = 100.00\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000-0}{4.9000}\right|\times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923-1.0000}{3.0923}\right|\times 100 = 67.662\%$$

The maximum percentage relative error after the first iteration is 100%

Gauss-Seidel Method

Example 2 – solution … cont.

Repeating more iterations, the following values are obtained

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.876
3	0.74275	80.236	3.1644	17.408	3.9708	4.0042
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$. Note that the exact solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

Gauss-Seidel Method

Example 3

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Gauss-Seidel Method

Example 3 – solution

Conducting six iterations, the following values are obtained

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	−196.15	110.71	14.421	94.453	−462.30	110.96
3	−1995.0	109.83	−116.02	112.43	4718.1	109.80
4	−20149	109.90	1204.6	109.63	−47636	109.90
5	2.0364×10^5	109.89	−12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^5	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

The values are not converging?

Gauss-Seidel Method

diagonally dominant matrix

Example 3 $$[A]=\begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

Example 2 $$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

If a system of linear equations is not diagonally dominant, check to see if rearranging the equations can form a diagonally dominant matrix.

Computer based solution method

- LSARG subroutine (Fortran IMSL subroutine that solves a real general system of linear equations with iterative refinement)

Solve a real general system of linear equations with iterative refinement.

Usage

CALL LSARG (N, A, LDA, B, IPATH, X)

Arguments

N — Number of equations. (Input)

A — N by N matrix containing the coefficients of the linear system. (Input)

LDA — Leading dimension of A exactly as specified in the dimension statement of the calling program. (Input)

B — Vector of length N containing the right-hand side of the linear system. (Input)

IPATH — Path indicator. (Input)
IPATH = 1 means the system *AX* = *B* is solved.
IPATH = 2 means the system *ATX* = *B* is solved.

X — Vector of length N containing the solution to the linear system. (Output)

Computer based solution method

- LSARG subroutine (Fortran IMSL subroutine that solves a real general system of linear equations with iterative refinement)

```
dimension A(4,4), b(4), x(4)

N=4
LDA=N

A(1,1)=1100
A(1,2)=0
A(1,3)=0
A(1,4)=0
A(2,1)=1000
A(2,2)=-1400
A(2,3)=100
A(2,4)=0
A(3,1)=0
A(3,2)=1100
A(3,3)=-1240
A(3,4)=100
A(4,1)=0
A(4,2)=0
A(4,3)=1100
A(4,4)=-1250

data b/1000.0,0.0,0.0,0.0/

call lsarg(N,A,LDA,B,1,X)


write(6,*)X

stop
end
```

Solution:

9.090909E-01 6.968810E-01 6.654245E-01 5.855736E-01

Stop - Program terminated.