Nonlinear Diffusion Approximations of Queuing Networks

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Dedicated to Professor Gopinath Kallianpur
on the Occasion of his 70th Birthday

Abstract

The aim of this paper is to sketch two examples of simple queuing networks that possess nonlinear diffusion approximations. These approximations are obtained by considering hydrodynamic limits for the corresponding interacting particle systems obeying various exclusion rules. Asymptotic analysis of the approximating nonlinear pdes is also provided.

1 Introduction

The method of (linear) diffusion approximations is well established in queuing theory and has become a standard textbook tool (see, e.g., Glynn

(1990)). The goal of the present paper is to sketch several examples of simple queuing networks that possess *nonlinear* diffusion approximations. These approximations are obtained by considering hydrodynamic limits for the corresponding interacting particle systems obeying various exclusion rules.

In Section 2 we recall an example of a queuing network consisting of infinitely many servers in series and the corresponding nearest neighbor exclusion particle systems. In this case the hydrodynamic limit gives rise to the nonlinear parabolic partial differential equation with quadratic inertial nonlinearity known as the Burgers equation. The example was considered by a number of authors (see, e.g., Kipnis (1986), Benassi and Fouque (1987), Srinivasan (1991,1993); see also Woyczynski (1998)) and it provided inspiration for the present work.

The following two sections describe two new classes of queuing networks for which the nonlinear diffusion approximation can be identified. The material here is preliminary and the final version of these results, including full proofs, will appear elsewhere.

More precisely, Section 3 provides a nonlinear diffusion approximation for what we call the *gossiping secretaries (GS) queuing network* where the servers are in series but each server does not start serving the customers until a certain minimal number of customers queues up in its queue. The mental picture is that of a secretary gossiping with the first arriving customer until the next customer arrives; only then the service begins. Several other practical interpretations in manufacturing, military command and control systems, and computer networks are possible. The nonlinear hyperbolic partial differential equations arising in this context have polynomial “intertial type” nonlinearities.

Section 4 studies a similar problem for multiserver queues. Although the nonlinear equations arising in this model are different from the ones arising in the GS model, the nonlinearities are also polynomial.

The equations that arose in Sections 3 and 4 are special cases of so-called conservation laws. For special initial data which are piecewise constant on halflines (the Riemann problem) and which, e.g., represent the queuing system empty at time $t = 0$, they can be solved explicitly. These shock and rarefaction wave solutions are discussed in Section 5.

For general initial data the above conservation laws cannot usually be found in closed terms. So, in Section 5, we briefly describe asymptotic properties of solutions of the nonlinear diffusion equations which are parabolic regularizations of hyperbolic equations that appeared in Sections 3 and 4. The method is based on subtle functional analytic methods developed for such (and more general, pseudodifferential) equations in Escobedo and Zuazua (1991), Escobedo, Velzquez and Zuazua (1993), and Biler, Karch and Woyczynski (1999, 2000). A Monte Carlo method for numerical handling of such equations is proposed in a separate paper by Margolius, Subramanian and Woyczynski (2000).

2 Servers in series and nearest neighbor asymetric exclusion particle systems

This is the example considered by Kipnis (1986), Benassi, Fouque (1987), and Srinivasan (1991,1993) which is recalled here for the sake of comparison with results of Sections 3 and 4 and to introduce the notation to be used later on. The starting point is the observation that the queuing system consisting of an infinite series of queues can be interpreted in the language of the one-dimensional nearest neighbor simple exclusion process (see, e.g., Liggett(1985)). Indeed, if the lattice location of the i-th particle is denoted by x_i then, in view of the exclusion dynamics and nearest neighbor jumps, at time t

$$\ldots < x_{-1}(t) < x_0(t) < x_1(t) < \ldots \tag{1}$$

Assume that the rate of this process is 1. If we denote by $\eta_i(t)$ the random variable equal to the number of empty sites between $x_i(t)$ and $x_{i+1}(t)$, then $\eta_i(t)$ can be considered as the length of the ith queue for an infinite queuing system with single servers in series, each with an exponential service time with intensity 1. Indeed, when the ith particle jumps to the right by one unit, then $\eta_i(t)$ changes into $\eta_i(t)-1$ which means that the service for one customer was completed at the ith server, and η_{i+1} is changed to $\eta_{i+i}(t)+1$, which means that a new customer was added to the queue at the $(i+1)$st server. In other words, the customer in the ith queue is served in exponential time with rate 1 and then joins the $(i-1)$st queue with probability p and $(i+1)$st queue with probability $1-p$.

Another way to code the asymmetric exclusion interacting particle system is by listing its states

$$X(t) = \{X(k,t) : t \geq 0, k \in \mathbf{Z}\} \in \{0,1\}^{\mathbf{Z}}, \tag{2}$$

The set $\{k : X(k,t) = 1\} \subset \mathbf{Z}$ is the set of occupied sites at time t. In the totally asymmetric case $p = 1$, the infinitesimal generator for the Markov process $X(t)$ (which does exist, see, e.g., Liggett (1985))

$$\mathcal{L}f(X) = \sum_{k\in Z} X(k)\big(1 - X(k+1)\big)\big[f(X^{k,k+1}) - f(X)\big], \tag{3}$$

where the state $X^{k,k+1}$ is obtained from the state X by setting $X(k) = 0, X(k+1) = 1$ and keeping the other values fixed.

The above system's dynamics can also be encoded in the infinite system of ordinary stochastic differential equations

$$\begin{aligned} dX(t,k) \quad &= \quad X(t^-,k-1)\big[1 - X(t^-,k)\big]\,dP(t,k-1) \\ &\quad\;\; -X(t^-,k)\big[1 - X(t^-,k+1)\big]\,dP(t,k) \end{aligned}$$

$$
\begin{aligned}
&+X(t^-,k+1)\big[1-X(t^-,k)\big]\,dQ(t,k+1)\\
&-X(t^-,k)\big[1-X(t^-,k-1)\big]\,dQ(t,k)
\end{aligned}
\tag{4}
$$

where $P(t,k)$ and $Q(t,k)$, $k \in \mathbf{Z}$, are independent Poisson processes with intensities p and $(1-p)$, representing jumps to the right and jumps to the left, respectively.

Define the hyperbolic rescalings

$$X^h(t,x) = \sum_{k\in\mathbf{Z}} X\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x), \tag{5}$$

$$P^h(t,x) = \sum_{k\in\mathbf{Z}} P\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x), \tag{6}$$

$$Q^h(t,x) = \sum_{k\in\mathbf{Z}} Q\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x), \tag{7}$$

and introduce notation

$$F_{\pm h}u(x) = u(x)\big(1-u(x\pm h)\big), \tag{8}$$

$$D_{\pm h}u(x) = \pm\frac{u(x\pm h)-u(x)}{h}. \tag{9}$$

A direct verification shows that the system (4) can now be written in the form

$$
\begin{aligned}
dX^h(t,x) \quad = \quad & -D_{-h}\big[F_h\big(X^h(t^-,x)\big)\,d(hP^h(t,x))\big]\\
& +D_h\big[F_{-h}\big(X^h(t^-,x)\big)\,d(hQ^h(t,x))\big].
\end{aligned}
\tag{10}
$$

THEOREM 2.1. *(Benassi and Fouque (1987)) Let $p \neq 1/2$. As $h \to 0$, the solution $X^h(t,x)\,dx$ of (10) converges weakly to $u(t,x)\,dx$, where $u(t,x)$ is a decreasing and right continuous in the x-variable weak solution of the nonlinear Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial B(u)}{\partial x} = 0, \tag{11}$$

$$u(0,x) = u_0(x) = b\mathbf{1}_{(-\infty,0]}(x) + a\mathbf{1}_{(0,\infty]}(x), \tag{12}$$

with some $0 \le a < b < \infty$ and

$$B(u) = u(1-u). \tag{13}$$

Moreover, for all t, x, we have $a \le u(t,x) \le b$.

Recall that the weak solution is understood in the following sense: For every smooth function $\phi : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ with compact support,

$$\int_{\mathbf{R}^+}\int_{\mathbf{R}} [u\phi_t + (2p-1)F(u)\phi_x]\;dx\,dt = -\int_{\mathbf{R}} u_0(x)\phi(0,x)\,dx. \tag{14}$$

Heuristically, the result is plausible, since, as $h \to 0$ in (10), $hP^h(t,x) \to pt$, $hQ^h(t,x) \to (1-p)t$, $D_{\pm h} \to \partial/\partial x$, and $F_{\pm h} \to F$.

Remark 2.1. For $p = 1/2$ it is easy to see that the hyperbolic scaling is inappropriate and the usual parabolic scaling leads to the standard linear diffusion equation.

Remark 2.2. Equation (11), after a linear transformation, can be changed into a classic hyperbolic Riemann equation $u_t + (u^2)_x = 0$, which after parabolic regularization becomes the Burgers equation $u_t + (u^2)_x = \nu u_{xx}$. For the theory of such equations see Smoller (1994), and Woyczynski (1998), in the case when random initial data are considered.

3 Gossiping secretaries (GS) network and asymmetric exclusion particle systems with nonlocal but finite-range interactions

In this section we will describe an extension of the queues-in-series/nearest-neighbor-asymmetric-exclusion-system model discussed in Section 2 in the sense that it will permit longer but still finite-range interactions. Consider an interacting particle system $X(t) = \{X(k,t) : t \geq 0, k \in \mathbf{Z}\} \in \{0,1\}^{\mathbf{Z}}$, operating under the following dynamics:

(i) There is always at most one particle per site;

(ii) Each particle has an exponential alarm clock with rate 1, the clocks are independent of each other. When the particle's clock rings the particle moves to the right (left) with probability p $(1-p)$ by one step but only if the site it is moving to is unoccupied *and* the site immediately to the right (left) of the target site is also unoccupied.

In the totally asymetric case $p = 1$, the model has an infinitesimal generator of the form

$$\mathcal{L}f(X) = \sum_{k \in Z} X(k)\big(1 - X(k+1)\big)\big(1 - X(k+2)\big)\big[f(X^{k,k+1}) - f(X)\big], \quad (15)$$

As before, we can interpret the above particle system as a queuing network of server in series with the number of empty sites between the particles i and $(i+1)$ (that is the interparticle distance minus one) being interpreted as the queue length at the ith server, but the regime we are now considering will activate the server only if the queue length is at least 2. In other words, the mental picture could be that of the network of secretaries in series, with each secretary gossiping with the first customer until another customer shows up and only then beginning to serve the first customer in exponential time.

In more generality, suppose the secretaries will not attend to a customer until there is a queue of m customers waiting. The system's dynamics can then be encoded in the infinite system of ordinary stochastic differential equations

$$\begin{aligned}
&dX(t,k)\\
&= X(t^-,k-1)\big[1-X(t^-,k)\big]\cdot\ldots\cdot\big[1-X(t^-,k+m-1)\big]\,dP(t,k-1)\\
&-X(t^-,k)\big[1-X(t^-,k+1)\big]\cdot\ldots\cdot\big[1-X(t^-,k+m)\big]\,dP(t,k)\\
&+X(t^-,k+1)\big[1-X(t^-,k)\big]\cdot\ldots\cdot\big[1-X(t^-,k-m+1)\big]\,dQ(t,k+1)\\
&-X(t^-,k)\big[1-X(t^-,k-1)\big]\cdot\ldots\cdot\big[1-X(t^-,k-m)\big]\,dQ(t,k), \qquad (16)
\end{aligned}$$

where $P(t,k)$ and $Q(t,k)$, $k\in\mathbf{Z}$, are independent Poisson processes with intensities p and $(1-p)$, representing jumps to the right and jumps to the left, respectively.

Define the hyperbolic rescalings $X^h(t,x), P^h(t,x)$, and $Q^h(t,x)$ as in (5-7) and the difference operator $D_{\pm h}u(x) =$ as in (9), but,

$$F_{\pm h}u(x) = u(x)\big(1-u(x\pm h)\big)\cdot\ldots\cdot\big(1-u(x\pm mh)\big), \qquad (17)$$

In this notation, one can verify directly that X_h satisifies the equation

$$\begin{aligned}
dX^h(t,x) \quad &= \quad -D_{-h}\big[F_h(X^h(t^-,x))\,d(hP^h(t,x))\big]\\
&\qquad +D_h\big[F_{-h}(X^h(t^-,x))\,d(hQ^h(t,x))\big]. \qquad (18)
\end{aligned}$$

Indeed, working backwards, beginning with (18) and substituting for $D_{\pm h}$ we obtain

$$\begin{aligned}
dX^h(t,x) \quad &= \quad \frac{1}{h}\left[F_h(X^h(t^-,x-h))d(hP^h(t,x-h))\right.\\
&\qquad \left.-F_h(X^h(t^-,x))d(hP^h(t,x))\right]\\
&\quad + \quad \frac{1}{h}\left[F_{-h}(X^h(t^-,x+h))d(hQ^h(t,x+h))\right.\\
&\qquad \left.-F_{-h}(X^h(t^-,x))d(hQ^h(t,x))\right]. \qquad (19)
\end{aligned}$$

Next, substituting (17) for $F_{\pm h}$, gives

$$\begin{aligned}
&dX^h(t,x) =\\
+\;& \frac{1}{h}\Big[X^h(t^-,x-h)\big[1-X^h(t^-,x)\big]\cdot\ldots\cdot\big[1-X^h(t^-,x+h(m-1))\big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times d(hP^h(t,x-h))\\
&-X^h(t^-,x)\big[1-X^h(t^-,x+h)\big]\cdot\ldots\cdot[1-X^h(t^-,x+mh)]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times d(hP^h(t,x))\Big]\\
+\;& \frac{1}{h}\Big[X^h(t^-,x+h)\big[1-X^h(t^-,x)\big]\cdot\ldots\cdot\big[1-X^h(t^-,x-(m-1)h)\big]
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times d(hQ^h(t,x+h)) \\
&-X^h(t^-,x)\big[1-X^h(t^-,x-h)\big]\cdot\ldots\cdot\big[1-X^h(t^-,x-mh)\big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times d(hQ^h(t,x))\Big]. \qquad (20)
\end{aligned}
$$

¿From the definition of $P^h(t,x)$, and $Q^h(t,x)$ we have

$$
\begin{aligned}
& dX^h(t,x) = \\
+\;& \frac{1}{h}\Big[X^h(t^-,x-h)\big[1-X^h(t^-,x)\big]\cdot\ldots\cdot\big[1-X^h(t^-,x+h(m-1))\big] \\
& \qquad\qquad\qquad \times d\Big(h\sum_{k\in\mathbf{Z}} P\Big(\frac{t}{h},k\Big)\,\mathbf{1}_{[hk,h(k+1))}(x-h)\Big) \\
& -X^h(t^-,x)\big[1-X^h(t^-,x+h)\big]\cdot\ldots\cdot\big[1-X^h(t^-,x+mh)\big] \\
& \qquad\qquad\qquad \times d\Big(h\sum_{k\in\mathbf{Z}} P\Big(\frac{t}{h},k\Big)\,\mathbf{1}_{[hk,h(k+1))}(x)\Big)\Big] \\
+\;& \frac{1}{h}\Big[X^h(t^-,x+h)\big[1-X^h(t^-,x)\big]\cdot\ldots\cdot\big[1-X^h(t^-,x-(m-1)h)\big] \\
& \qquad\qquad\qquad \times d\Big(h\sum_{k\in\mathbf{Z}} Q\Big(\frac{t}{h},k\Big)\,\mathbf{1}_{[hk,h(k+1))}(x+h)\Big) \\
& -X^h(t^-,x)\big[1-X^h(t^-,x-h)\big]\cdot\ldots\cdot\big[1-X^h(t^-,x-mh)\big] \\
& \qquad\qquad\qquad \times d\Big(h\sum_{k\in\mathbf{Z}} Q\Big(\frac{t}{h},k\Big)\,\mathbf{1}_{[hk,h(k+1))}(x)\Big)\Big].
\end{aligned} \tag{21}
$$

The definition of $X^h(t,x)$ and substitution $x=hj$ gives

$$
\begin{aligned}
& dX^h(t,hj) = \\
+\;& \frac{1}{h}\Big[X\Big(\frac{t}{h},j-1\Big)\Big(1-X\Big(\frac{t}{h},j\Big)\Big)\cdot\ldots\cdot\Big(1-X\Big(\frac{t}{h},j+m-1\Big)\Big) \\
& \qquad\qquad\qquad \times d\Big(hP\Big(\frac{t}{h},j-1\Big)\Big) \\
& -X\Big(\frac{t}{h},j\Big)\Big(1-X\Big(\frac{t}{h},j+1\Big)\Big)\cdot\ldots\cdot\Big(1-X\Big(\frac{t}{h},j+m\Big)\Big) \\
& \qquad\qquad\qquad \times d\Big(hP\Big(\frac{t}{h},j\Big)\Big)\Big] \\
+\;& \frac{1}{h}\Big[X\Big(\frac{t}{h},j+1\Big)\Big(1-X\Big(\frac{t}{h},j\Big)\Big)\cdot\ldots\cdot\Big(1-X\Big(\frac{t}{h},j-(m-1)\Big)\Big) \\
& \qquad\qquad\qquad \times d\Big(hQ\Big(\frac{t}{h},j+1\Big)\Big) \\
& -X\Big(\frac{t}{h},j\Big)\Big(1-X\Big(\frac{t}{h},j-1\Big)\Big)\cdot\ldots\cdot\Big(1-X\Big(\frac{t}{h},j-m\Big)\Big) \\
& \qquad\qquad\qquad \times d\Big(hQ\Big(\frac{t}{h},j\Big)\Big)\Big],
\end{aligned} \tag{22}
$$

so that, finally,

$$
\begin{aligned}
& dX\left(\frac{t}{h}, j\right) = \\
& \left[X\left(\frac{t}{h}, j-1\right)\left(1-X\left(\frac{t}{h}, j\right)\right) \dots \left(1-X\left(\frac{t}{h}, j+m-1\right)\right)\right. \\
& \qquad\qquad\qquad\qquad \times d\left(P\left(\frac{t}{h}, j-1\right)\right) \\
& -X\left(\frac{t}{h}, j\right)\left(1-X\left(\frac{t}{h}, j+1\right)\right) \dots \left(1-X\left(\frac{t}{h}, j+m\right)\right) \\
& \qquad\qquad\qquad\qquad \left.\times d\left(P\left(\frac{t}{h}, j\right)\right)\right] \\
+ \; & \left[X\left(\frac{t}{h}, j+1\right)\left(1-X\left(\frac{t}{h}, j\right)\right) \dots \left(1-X\left(\frac{t}{h}, j-(m-1)\right)\right)\right. \\
& \qquad\qquad\qquad\qquad \times d\left(Q\left(\frac{t}{h}, j+1\right)\right) \\
& -X\left(\frac{t}{h}, j\right)\left(1-X\left(\frac{t}{h}, j-1\right)\right) \dots \left(1-X\left(\frac{t}{h}, j-m\right)\right) \\
& \qquad\qquad\qquad\qquad \left.\times d\left(Q\left(\frac{t}{h}, j\right)\right)\right],
\end{aligned} \tag{23}
$$

which justifies (19).

Now note that in view of (19), heuristically, it is clear that

$$
\begin{aligned}
\lim_{h\to 0} dX^h(t,x) = & - \frac{\partial}{\partial x}\Big[u(t,x)(1-u(t,x))^m\Big]\, d(pt) \\
& + \frac{\partial}{\partial x}\Big[u(t,x)(1-u(t,x))^m\Big]\, d((1-p)t),
\end{aligned} \tag{24}
$$

which provides an argument in support of the following result about approximation of density profiles for the GS queuing networks by solutions of a nonlinear hyperbolic equation.

THEOREM 3.1. *Let $p \neq 1/2$. As $h \to 0$, the solution $X^h(t,x)\,dx$ of (18) converges weakly to $u(t,x)\,dx$, where $u(t,x)$ is a weak solution of the nonlinear Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial F(u)}{\partial x} = 0. \tag{25}$$

$$u(0,x) = u_0(x) = b\mathbf{1}_{(-\infty,0]}(x) + a\mathbf{1}_{(0,\infty)}(x), \tag{26}$$

with some $0 \le a < b < \infty$ and

$$F(u) = [u(t,x)(1-u(t,x))^m]. \tag{27}$$

Moreover, for all t, x, we have $a \le u(t,x) \le b$.

4 Networks with multiserver nodes and particle systems with state-dependent rates

The exclusion interacting particle system $X(t) = \{X(t,k) : t \geq 0, k \in \mathbf{Z}\} \in 2^{\mathbf{Z}}$, discussed in this section is operating under the following dynamics:

(i) There is always at most one particle per site;

(ii) Each particle has an exponential alarm clock with state-dependent rate which also depends on the constants $0 \leq p \leq 1$, and a positive integer N. The rate is p times the minimum of r, the number of empty sites to the right, and N, plus $1-p$ times the minimum of l, the number of empty sites to the left, and N. When the particle's clock rings, the particle moves to the right by one step with probability

$$p_{r,l} = \frac{p\min\{N,r\}}{p\min\{N,r\} + (1-p)\min\{N,l\}},$$

but only if the target site to the right is unoccupied. The particle moves to the left with probability

$$1 - p_{r,l} = \frac{(1-p)\min\{N,l\}}{p\min\{N,r\} + (1-p)\min\{N,l\}},$$

but only if the target site to the left is unoccupied.

The queuing-theoretic interpretation of this model is analogous to the examples in Sections 2-3. We have an infinite series of queues. As in the nearest neighbor exclusion system of Section 2, we denote by $\eta_i(t)$ the random variable equal to the number of empty sites between $x_i(t)$ and $x_{i+1}(t)$, where $x_i(t)$ is the location of the ith particle, and, as before, $\eta_i(t)$ represents the length of the ith queue. In this model though, we have N identical servers at each particle. When the servers are serving the queue to the right, service for the first customer served is completed at the rate $\min\{N,r\}$. When the servers are serving the queue to the left, service for the first customer served is completed at the rate $\min\{N,l\}$. The former event occurs with probability $p_{r,l}$, and the latter with probability $1-p_{r,l}$. When the ith particle jumps to the right, $\eta_i(t)$ changes to $\eta_i(t)-1$ which means that service for one customer was completed at the ith server, and $\eta_{i+1}(t)$ is changed to $\eta_{i+1}(t)+1$, i.e. a new customer is added to the $(i+1)$st queue. When $p=1$, customers move through the queueing stations sequentially. When $0 < p < 1$, customers can move back and forth through the queues in the network. When there are fewer in the queue than the number of servers then this regime corresponds to all customers being in service and the jump occurs when the first of the $\eta_i(t)$ customers has completed service.

The corresponding infinitesimal generator for this regime in the totally asymmetric case is

$$Lf(X) = \sum_{k\in\mathbf{Z}} \left(X(t^-,k) \left[\sum_{m=1}^{N-1} m \left(\prod_{i=1}^{m} [1 - X(t^-,k+i)] \right) X(t^-,k+m+1) + N \left(\prod_{i=1}^{N} [1 - X(t^-,k+i)] \right) \right] \right) [f(X^{k,k+1}) - f(X)]. \tag{28}$$

We will now describe the evolution of this particle system via a system of stochastic differential equations. Note that if the site to the left of site k is occupied with particle i and $\eta_i = 1$, then a jump to the right occurs according to the independent Poisson process $P(t,k)$ with intensity p. This event occurs if $X(t^-,k-1)(1-X(t^-,k))X(t^-,k+1) = 1$. More generally, if the site to the left of site k is occupied with particle i and $\eta_i = r < N$, then a jump to the right occurs according to the independent Poisson process $P_r(t,k)$ with intensity rp. This event occurs if $X(t^-,k-1)(1-X(t^-,k))(1-X(t^-,k+1))\cdots(1-X(t^-,k+r-1))X(t^-,k+r) = 1$. If the site to the left of k is occupied with particle i and $\eta_i \geq N$, then the quantity $X(t^-,k-1)(1-X(t^-,k))(1-X(t^-,k+1))\cdots(1-X(t^-,k+N)) = 1$ and a jump to the right is governed by the independent Poisson process $P_N(t,k)$ with intensity Np.

Jumps from the site k are governed by the same rules so that if $X(t^-,k)(1-X(t^-,k+1))X(t^-,k+2) = 1$ a jump to the right out of site k occurs according to the independent Poisson process $P(t,k)$ with intensity p. More generally, if site k is occupied with particle i and $\eta_i = r < N$, then a jump to the right occurs according to the independent Poisson process $P_r(t,k)$ with intensity rp. This event occurs if $X(t^-,k-1)(1-X(t^-,k))(1-X(t^-,k+1))\cdots(1-X(t^-,k+r-1))X(t^-,k+r) = 1$. If the site to the left of k is occupied with particle i and $\eta_i \geq N$, then the quantity $X(t^-,k-1)(1-X(t^-,k))(1-X(t^-,k+1))\cdots(1-X(t^-,k+N)) = 1$ and a jump to the right is governed by the independent Poisson process $P_N(t,k)$ with intensity Np. For jumps to the left, we replace $P(t,k)$ with $Q(t,k)$ and p with $(1-p)$ and change signs, but otherwise the analysis is the same.

This analysis leads to the following system ($k \in \mathbf{Z}$) of ordinary stochastic differential equations:

$$\begin{aligned} & dX(t,k) = \\ + \quad & X(t^-,k-1) \left[\sum_{m=1}^{N-1} m \left(\prod_{i=1}^{m} [1 - X(t^-,k+(i-1))] \right) X(t^-,k+m) \right. \\ & \left. + N \left(\prod_{i=1}^{N} [1 - X(t^-,k+(i-1))] \right) \right] dP(t,k-1) \end{aligned}$$

$$
\begin{aligned}
- \quad & X(t^-,k)\left[\sum_{m=1}^{N-1} m\left(\prod_{i=1}^{m}[1-X(t^-,k+i)]\right)X(t^-,k+m+1)\right.\\
& \left. +N\left(\prod_{i=1}^{N}[1-X(t^-,k+i)]\right)\right] dP(t,k)\\
+ \quad & X(t^-,k+1)\left[\sum_{m=1}^{N-1} m\left(\prod_{i=1}^{m}[1-X(t^-,k-(i-1))]\right)X(t^-,k-m)\right.\\
& \left. +N\left(\prod_{i=1}^{N}[1-X(t^-,k-(i-1))]\right)\right] dQ(t,k+1)\\
- \quad & X(t^-,k)\left[\sum_{m=1}^{N-1} m\left(\prod_{i=1}^{m}[1-X(t^-,k-i)]\right)X(t^-,k-m-1)\right.\\
& \left. +N\left(\prod_{i=1}^{N}[1-X(t^-,k-i)]\right)\right] dQ(t,k), \qquad (29)
\end{aligned}
$$

After the hyperbolic rescaling (5-7) the system can be written as a single equation

$$
\begin{aligned}
dX^h(t,x) \quad = \quad & -D_{-h}\big[G_h\big(X^h(t^-,x)\big)\, d\big(hP^h(t,x)\big)\big]\\
& +D_h\big[G_{-h}\big(X^h(t^-,x)\big)\, d\big(hQ^h(t,x)\big)\big], \qquad (30)
\end{aligned}
$$

where

$$
\begin{aligned}
G_{\pm h}u(x) \quad = \quad & \sum_{m=1}^{N-1} m u(x)\left(\prod_{i=1}^{m}[1-u(x\pm ih)]\right)u(x\pm(m+1)h)\\
& +Nu(x)\left(\prod_{i=1}^{N}[1-u(x\pm ih)]\right), \qquad (31)
\end{aligned}
$$

and $D_{\pm h}$ is as defined above in (9).

Indeed, substituting for $D_{\pm h}$ and $G_{\pm h}$ in (30) we obtain

$$
\begin{aligned}
& dX^h(t,x) =\\
+ \quad & \frac{1}{h}\left\{\left[\sum_{m=1}^{N-1} mX^h(t^-,x-h)\left(\prod_{i=1}^{m}(1-X^h(t^-,x+(i-1)h))\right)\right.\right.\\
& \qquad \times X^h(t^-,x+mh)\\
+ \quad & \left. NX^h(t^-,x-h)\left(\prod_{i=1}^{N}(1-X^h(t^-,x+(i-1)h))\right)\right] d(hP^h(t^-,x-h))\\
- \quad & \left[\sum_{m=1}^{N-1} mX^h(t^-,x)\left(\prod_{i=1}^{m}(1-X^h(t^-,x+ih))\right)X^h(t^-,x+(m+1)h)\right.\\
+ \quad & \left.\left. NX^h(t^-,x)\left(\prod_{i=1}^{N}(1-X^h(t^-,x+ih))\right)\right] d(hP^h(t^-,x))\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{h}\left\{\left[\sum_{m=1}^{N-1} mX^h(t^-,x+h)\left(\prod_{i=1}^{m}(1-X^h(t^-,x-(i-1)h))\right)\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times X^h(t^-,x-mh)\\
&+ \quad NX^h(t^-,x+h)\left(\prod_{i=1}^{N}(1-X^h(t^-,x-(i-1)h))\right)\bigg]\, d(hQ^h(t^-,x+h))\\
&- \quad \left[\sum_{m=1}^{N-1} mX^h(t^-,x)\left(\prod_{i=1}^{m}(1-X^h(t^-,x-ih))\right)X^h(t^-,x-(m+1)h)\right.\\
&+ \quad \left.\left. NX^h(t^-,x)\left(\prod_{i=1}^{N}(1-X^h(t^-,x-ih))\right)\right]d(hQ^h(t^-,x))\right\}. \qquad (32)
\end{aligned}
$$

From the definition of $P^h(t,x)$, and $Q^h(t,x)$,

$$
\begin{aligned}
& dX^h(t,x) = \\
&+ \quad \frac{1}{h}\left\{\left[\sum_{m=1}^{N-1} mX^h(t^-,x-h)\left(\prod_{i=1}^{m}[1-X^h(t^-,x+(i-1)h)]\right)\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times X^h(t^-,x+mh)\\
&+ \quad NX^h(t^-,x-h)\left(\prod_{i=1}^{N}[1-X^h(t^-,x+(i-1)h)]\right)\bigg]\\
&\qquad\qquad\qquad\qquad\qquad \times d\left(h\sum_{k\in\mathbf{Z}} P\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x-h)\right)\\
&- \quad \left[\sum_{m=1}^{N-1} mX^h(t^-,x)\left(\prod_{i=1}^{m}[1-X^h(t^-,x+ih)]\right)X^h(t^-,x+(m+1)h)\right.\\
&+ \quad NX^h(t^-,x)\left(\prod_{i=1}^{N}[1-X^h(t^-,x+ih)]\right)\bigg]\\
&\qquad\qquad\qquad\qquad\qquad \left.\times d\left(h\sum_{k\in\mathbf{Z}} P\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x)\right)\right\}\\
&+ \quad \frac{1}{h}\left\{\left[\sum_{m=1}^{N-1} mX^h(t^-,x+h)\left(\prod_{i=1}^{m}[1-X^h(t^-,x-(i-1)h)]\right)\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times X^h(t^-,x-mh)\\
&+ \quad NX^h(t^-,x+h)\left(\prod_{i=1}^{N}[1-X^h(t^-,x-(i-1)h)]\right)\bigg]\\
&\qquad\qquad\qquad\qquad\qquad \times d\left(h\sum_{k\in\mathbf{Z}} Q\left(\frac{t}{h},k\right)\mathbf{1}_{[hk,h(k+1))}(x+h)\right)\\
&- \quad \left[\sum_{m=1}^{N-1} mX^h(t^-,x)\left(\prod_{i=1}^{m}[1-X^h(t^-,x-ih)]\right)X^h(t^-,x-(m+1)h)\right.
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & N X^h(t^-, x) \left(\prod_{i=1}^{N} [1 - X^h(t^-, x - ih)] \right) \Bigg] \\
& \qquad \qquad \times d\left(h \sum_{k \in \mathbf{Z}} Q\left(\frac{t}{h}, k \right) \mathbf{1}_{[hk, h(k+1))}(x) \right) \Bigg\}.
\end{aligned}
\tag{33}
$$

Finally, utilizing the definition of $X^h(t,x)$ and substituting $x = hj$ gives

$$
\begin{aligned}
& dX^h(t,x) = \\
+ \quad & \Bigg[\sum_{m=1}^{N-1} m X\left(\frac{t}{h}, j-1\right) \left(\prod_{i=1}^{m} [1 - X\left(\frac{t}{h}, j+i-1\right)] \right) X\left(\frac{t}{h}, j+m\right) \\
+ \quad & N X\left(\frac{t}{h}, j-1\right) \left(\prod_{i=1}^{N} [1 - X\left(\frac{t}{h}, j+i-1\right)] \right) \Bigg] d\left(P\left(\frac{t}{h}, j-1\right) \right) \\
- \quad & \Bigg[\sum_{m=1}^{N-1} m X\left(\frac{t}{h}, j\right) \left(\prod_{i=1}^{m} [1 - X\left(\frac{t}{h}, j+i\right) \right) X\left(\frac{t}{h}, j+m+1\right) \\
+ \quad & N X\left(\frac{t}{h}, j\right) \left(\prod_{i=1}^{N} [1 - X\left(\frac{t}{h}, j+i\right)] \right) \Bigg] d\left(P\left(\frac{t}{h}, j\right) \right) \\
+ \quad & \Bigg[\sum_{m=1}^{N-1} m X\left(\frac{t}{h}, j+1\right) \left(\prod_{i=1}^{m} [1 - X\left(\frac{t}{h}, j-(i-1)\right)] \right) X\left(\frac{t}{h}, j-m\right) \\
+ \quad & N X\left(\frac{t}{h}, j+1\right) \left(\prod_{i=1}^{N} [1 - X\left(\frac{t}{h}, j-i+1\right)] \right) \Bigg] d\left(Q\left(\frac{t}{h}, j+1\right) \right) \\
- \quad & \Bigg[\sum_{m=1}^{N-1} m X\left(\frac{t}{h}, j\right) \left(\prod_{i=1}^{m} [1 - X\left(\frac{t}{h}, j-i\right)] \right) X\left(\frac{t}{h}, j-m-1\right) \\
+ \quad & N X\left(\frac{t}{h}, j\right) \left(\prod_{i=1}^{N} [1 - X\left(\frac{t}{h}, j-i\right)] \right) \Bigg] d\left(Q\left(\frac{t}{h}, j\right) \right).
\end{aligned}
\tag{34}
$$

This immediately yields the system (29).

As $h \to 0$, the equation (30) converges to the deterministic partial differential equation:

$$
\frac{\partial u}{\partial t} = (1-2p) \frac{\partial}{\partial x} \left[\sum_{m=1}^{N-1} m u(t,x)^2 \big(1 - u(t,x)\big)^m + N u(t,x) \big(1 - u(t,x)\big)^N \right].
\tag{35}
$$

Since

$$
\sum_{m=1}^{N-1} m u^2 (1-u)^m + N u (1-u)^N
$$

$$\begin{aligned}
&= u^2(1-u)\left[-\frac{\partial}{\partial u}\sum_{m=0}^{N-1}(1-u)^m\right] + Nu(1-u)^N \\
&= u^2(1-u)\frac{\partial}{\partial u}\left[\frac{-1}{u}\big(1-(1-u)^N\big)\right] + Nu(1-u)^N \\
&= u^2(1-u)\left[u^{-2}\big(1-(1-u)^N\big) - \frac{N}{u}(1-u)^{N-1}\right] + Nu(1-u)^N \\
&= (1-u)\big(1-(1-u)^N\big) - Nu(1-u)^N + Nu(1-u)^N \\
&= (1-u)-(1-u)^{N+1}. \qquad (36)
\end{aligned}$$

The above arguments lead to the following result.

Theorem 4.1. *Let $p \neq 1/2$. As $h \to 0$, the solution $X^h(t,x)\,dx$ of (30) converges weakly to $u(t,x)\,dx$, where $u(t,x)$ is a weak solution of the nonlinear Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial G(u)}{\partial x} = 0. \qquad (37)$$

$$u(0,x) = u_0(x) = b\mathbf{1}_{(-\infty,0]}(x) + a\mathbf{1}_{(0,\infty)}(x), \qquad (38)$$

with some $0 \le a < b < \infty$ and

$$G(u) = \Big[\big(1-u(t,x)\big) - \big(1-u(t,x)\big)^{N+1}\Big]. \qquad (39)$$

Moreover, for all t, x, we have $a \le u(t,x) \le b$.

5 Shock and rarefaction wave solutions for the Riemann problem for conservation laws

The nonlinear hyperbolic equations (25) and (37) describing the density profiles for the queuing networks in Sections 3 and 4 are special cases of general conservation laws (see, e.g., Smoller (1994)) of the form

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0. \qquad (40)$$

and in the case of initial conditions of the form

$$u(0,x) = u_0(x) = u_l\mathbf{1}_{(-\infty,0]}(x) + u_r\mathbf{1}_{(0,\infty]}(x), \qquad (41)$$

where u_l and u_r are constants (so called Riemann problem), they can be solved explicitly under some extra conditions on function H.

Let us recall (see, e.g., Smoller (1994)) that a bounded and measurable function $u(t,x)$ is called a (weak) solution of the initial-value problem

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0, \qquad u(0,x) = u_0(x), \tag{42}$$

with bounded and measurable initial data u_0 if

$$\int_{t\geq 0}\int_{\mathbf{R}} \bigl(u\phi_t + H(u)\phi_x\bigr)\,dx\,dt + \int_{t=0} u_0\phi\,dx = 0 \tag{43}$$

In general, solutions are not unique unless additional assumptions, such as the entropy condition mentioned below, are satisfied.

The solutions of the Riemann problem (40-41) are obviously invariant under hyperbolic rescaling, that is, for every constant $\lambda > 0$

$$u_\lambda(t,x) = u(\lambda t, \lambda x)$$

is a solution whenever u is. Thus one looks for the solutions of the form

$$u(t,x) = v(x/t) \tag{44}$$

This gives rise to three types of local behavior of the solutions of u:

- $u(t,x)$ is constant;
- $u(t,x)$ is a *shock wave* of the form

  $$u(t,x) = u_0\mathbf{1}_{(-\infty,Vt)}(x) + u_1\mathbf{1}_{[Vt,\infty)}(x), \tag{45}$$

  traveling with the velocity

  $$V = \frac{H(u_0) - H(u_1)}{u_0 - u_1}.$$

  For the sake of uniqueness one adds here the entropy condition $H'(u_0) > V > H'(u_1)$.
- $u(t,x)$ is a continuous *rarefaction wave* of the form (44) where v satisfies the ordinary differential equation

  $$v'(\xi)\bigl(H'(v(\xi)) - \xi\bigr) = 0. \tag{46}$$

We will apply the above standard observations to the case of the GS and multiple servers queuing networks starting with the latter because it is somewhat simpler (see also Margolius (1999) for other approaches to multiserver queues). The animated graphics files depicting the time evolution of density profiles for both types of networks can be seen at our website: `http://www.academic.csuohio.edu/bmargolius/waves/wavesa.htm`.

Density profiles for the multiple servers network. The network was described in Section 4 and we will study it here in the special case of totally asymmetric $p = 1$ and the initial condition (41) where $u_l = 1$ and $u_r = 0$; other situations with Riemann-type data can treated in a similar fashion and will be analyzed elsewhere. So, from now on in this subsection

$$H(u) = \Big[\big(1-u\big) - \big(1-u\big)^{N+1}\Big]. \tag{47}$$

Thus, for ξ such that $v'(\xi) \neq 0$, the equation (46) can be written in the form

$$-1 + (N+1)(1-v)^N = \xi, \tag{48}$$

so that the solution of the Riemann problem is

$$u(t,x) = \begin{cases} 1, & \text{for } x < -t; \\ 1 - [(x/t+1)/(N+1)]^{1/N}, & \text{for } -t \leq x < Nt; \\ 0, & \text{for } Nt \leq x. \end{cases} \tag{49}$$

Fig. 5.1. The evolution of the density profiles for the multiserver network described in Section 4 in the case of $N = 1$. This is the network considered by Benassi and Fouque (1987) and the Burgers equation in the hydrodynamic limit.

Fig. 5.2. The evolution of the density profiles for the multiserver network described in Section 4 in the case of $N = 4$.

Fig. 5.3. The evolution of the density profiles for the multiserver network described in Section 4 in the case of $N = 100$.

Density profiles for the GS network. This case is slightly more difficult as uniqueness questions arise because of the bifurcations. For the GS network, we examine the solution with initial condition $u_l = 2/(m+1)$, $u_r = 0$, that is, we begin with an average of $m-1$ customers for every 2 servers to the left of zero and no servers to the right of zero. Proceeding as in the multiserver case, we have

$$H(u) = u(1-u)^m, \tag{50}$$

so for ξ such that $v(\xi) \neq 0$, the equation (46) can be written in the form

$$(1-u)^{m-1}(1-(m+1)u) = \xi. \tag{51}$$

So, for the GS network the solution of the Riemann problem bifurcates at (x_b, u_b), where

$$\begin{aligned} x_b &= -t\left(\frac{m-1}{m+1}\right)^{m-1}, \\ u_b &= 2/(m+1). \end{aligned} \tag{52}$$

The figure below illustrates the case where $m = 4$, but the solutions are similar to those shown for $m > 1$. When $m = 1$ we have a simple asymmetric exclusion process.

Fig. 5.4. The evolution of the density profiles for the GS network for $m = 4$. The solution is not unique; a bifurcation is shown.

The lower branch is consistent with the entropy condition (see, e.g., Smoller (1994), p.). Define $g(u) = t(1-u)^{m-1}(1-(m+1)u)$ for $0 < u \leq 2/(m+1)$. Hence, in the case of a GS network, the solution of the Riemann problem satisfying the entropy condition is

$$u(t,x) = \begin{cases} 2/(m+1) & \text{for } x < -t\left(\frac{m-1}{m+1}\right)^{m-1}; \\ g^{-1}(x) & \text{for } -t\left(\frac{m-1}{m+1}\right)^{m-1} \leq x < t; \\ 0 & \text{for } t \leq x. \end{cases} \tag{53}$$

Note that at location $x = 0$, for $t > 0$, there are an average of m customers for every one server, i.e. $u(t,0) = 1/(m+1), \forall t > 0$; for $x < 0$ the average server does not have enough customers waiting to begin service, and for $x > 0$ the average server has a queue longer than the m customers required to begin service. We will analyse the behavior of the GS network more thoroughly elsewhere.

6 Nonlinear diffusion approximations

For initial conditions not of Riemann type, in particular those with integrable data, or for more general random initial conditions, obtaining solutions of the conservations law is not a simple matter, even in approximate fashion. The usual approach then is to consider a parabolic regularization (the viscosity method) by considering the *nonlinear diffusion* equations

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = \epsilon \mathcal{L} u, \qquad u(0,x) = u_0(x), \tag{54}$$

where $\mathcal{L}$ is a dissipative operator of elliptic type, like e.g. the Laplacian. Then, of course, with the exception of the quadratic case giving rise to the Burgers equation, one can not count on finding explicit solutions but two types of asymptotic results can be used as approximations.

The first kind provides the large time asymptotics of the regularized conservation laws and the second kind gives a Monte Carlo method of solving them via the interacting diffusions scheme (so-called propagation of chaos). We will briefly describe the two approaches.

Asymptotics for nonlinear diffusion equations. Not surprisingly, given the decay of their solution in time, the large time asymptotic behavior for parabolically regularized conservation laws is dictated by the asymptotic behavior of the nonlinearity $H(u)$ at points where the function is small. So, we have the following asymptotic results for regularized versions of (25) and (37):

THEOREM 6.1. *Let $\epsilon > 0, m \geq 1$ and $u(t,x)$ be a positive weak solution of the Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial F(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0,x) = u_0(x) \geq 0, \tag{55}$$

with $F(u) = [u(1-u)^m]$. Then

(i) If $u_0 \in L^1(\mathbf{R})$ then u has the same large time asymptotics as the solution of the linear diffusion equation

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0,x) = u_0(x) \geq 0, \tag{56}$$

or more precisely

$$\|u(t,x) - U(t,x)\|_1 \to 0 \quad \text{as} \quad t \to \infty, \tag{57}$$

where $U(t,x) = (g * u_0)(t, x - (2p-1)t)$ and $g(t,x) = (4\pi t)^{-1/2} \exp(-|x|^2/(4t))$ is the standard Gaussian kernel.

(ii) If $1 - u_0 \in L^1(\mathbf{R})$ then:

In the case $m = 1$, u has the same large time asymptotics (56-57) as the solution of the linear diffusion equation.

In the case $m = 2$, u has the same large time asymptotics as the self-similar source solution of the Burgers equations or more precisely, for each $p > 1$

$$t^{(1-1/p)/2}\|u(t,x) - U_M(t,x)\|_p \to 0 \quad \text{as} \quad t \to \infty, \tag{58}$$

where

$$U_M(t,x) = t^{-1/2}\exp(-x^2/(4t))\Big(K(M) + \frac{1}{2}\int_0^{x/(2\sqrt{t}} \exp(-\xi^2/4)\, d\xi\Big)^{-1}, \tag{59}$$

and $U_M(t,x) \to M\delta(x)$ as $t \to 0$ with $M = \|u_0\|_1$.

In the case $m \geq 3$, u has the same large time asymptotics as the solution of the heat equations or more precisely, for each $p > 1$ there exists a constant C such that

$$\|u(t,x) - U(t,x)\|_p \leq Ct^{-(1-1/p)/2}, \tag{60}$$

where $U(t,x) = (g * u_0)(t,x)$.

Sketch of the Proof. By the results of Escobedo and Zuazua (1991), Escobedo, Velazquez and Zuazua (1993) (see also Biler, Karch and Woyczynski (1999) for other regularizations of conservation laws) the asymptotic behavior of the solutions of the conservation laws (54) depends on the asymptotic behavior of the nonlinearity H at its small values. So, for $H(u) = (2p-1)F(u) = (2p-1)u(1-u)^m$,

$$\lim_{u \to 0} \frac{(2p-1)F(u)}{u} = 2p-1, \tag{61}$$

and

$$\lim_{u\to 1} \frac{(2p-1)F(u)}{(1-u)^m} = 2p-1. \tag{62}$$

The first condition (61), together with the standard step removing the drift term in the linear diffusion equation gives (i), and the case $m = 1$ in the second condition (62) gives the first part of (ii).

The critical case $m = 2$ in (62) yields the Burgers equation type asymptotics claimed in the second part of (ii), and the supercritical case $m \geq 3$ where the effect of the nonlinear convection term disappears in the limit.

THEOREM 6.2. *Let $\epsilon > 0, N \geq 1$ and $u(t, x)$ be a positive weak solution of the Cauchy problem*

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial G(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0, x) = u_0(x) \geq 0, \tag{63}$$

with $G(u) = (1-u) - (1-u)^{N+1}$. Then if either $u_0 \in L^1(\mathbf{R})$ or $1 - u_0 \in L^1(\mathbf{R})$ then u has the same large time asymptotics as the solution of the linear diffusion equation

$$\frac{\partial u}{\partial t} + (2p-1)\frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \qquad 1 \geq u(0, x) = u_0(x) \geq 0, \tag{64}$$

or more precisely

$$\|u(t, x) - U(t, x)\|_1 \to 0 \quad \text{as} \quad t \to \infty, \tag{65}$$

where $U(t, x) = (g * u_0)(t, x - (2p-1)t)$ and $g(t, x) = (4\pi t)^{-1/2} \exp(-|x|^2/(4t))$ is the standard Gaussian kernel.

Sketch of the Proof: The proof of this result relies on the same asymptotics results that were employed in the proof of Theorem 6.1. But in this case $H(u) = (2p-1)[(1-u) - (1-u)^{N+1}]$ which has the linear asymptotics at both $u = 0$ and $u = 1$. So, the result follows by the usual reduction to the heat equation.

Interacting diffusions approximations for nonlinear diffusion equations. This section discusses a possibility of a Monte Carlo type approximation for solutions of nonlinear diffusion equations of the type that arise as parabolic regularizations (54) of conservation laws of the encountered in Theorems 3.1, 4.1 and 4.2. The idea is to use the following scheme known as the propagation of chaos result and depends on the the construction of the so-called nonlinear McKean process for our equations.

The basic observation is that if the regularizing operator $\mathcal{L}$ is the infinitesimal generator of a Lévy process then the equation (54) (say, $\epsilon = 1$) can be formally interpreted as a "Fokker–Planck–Kolmogorov equation" for a "nonlinear" diffusion process in the McKean's sense. Indeed, consider a Markov process $X(t)$, $t \geq 0$, which is a solution of the stochastic differential equation

$$\begin{aligned} dX(t) &= dS(t) - u^{-1}H(u(X(t),t))\,dt, \\ X(0) &\sim u_0(x)\,dx \quad \text{in law,} \end{aligned} \tag{66}$$

where $S(t)$ is the Lévy process with generator $-\mathcal{L}$. Assuming that $X(t)$ is a unique solution of (66), we see that the measure-valued function $v(dx,t) = P(X(t) \in dx)$ satisfies the weak forward equation

$$\begin{aligned} \frac{d}{dt}\langle v(t), \eta\rangle &= \langle v(t), \widetilde{\mathcal{L}}_{u(t)}\eta\rangle, \ \eta \in \mathcal{S}(I\!R^n), \\ v(0) &= u(x,0)\,dx \end{aligned} \tag{67}$$

with $\widetilde{\mathcal{L}}_u = -\mathcal{L} + u^{-1}H(u) \cdot \nabla$. On the other hand $u(dx,t) = u(x,t)\,dx$ also solves (67) since

$$\frac{d}{dt}\langle u(t), \eta\rangle = \langle -\mathcal{L}u - \nabla \cdot H(u), \eta\rangle = \langle u, (-\mathcal{L} + u^{-1}H(u) \cdot \nabla)\eta\rangle$$

so that $v(dx,t) = u(dx,t)$ and, by uniqueness, u is the density of the solution of (66).

The above construction makes possible approximation of solutions of equations of the form (54) via finite systems of interacting diffusions. To illustrate our point we will formulate this Monte Carlo algorithm in the special, and well known Burgers equation case where $\mathcal{L} = \Delta$, is the usual Laplacian and the nonlinearity $H(x) = x^2$ is quadratic. The more general results needed for the analysis of GS and multiserver queuing networks are under development (see Calderoni and Pulvirenti (1983), Sznitman (1991), Zhang (1995), Funaki and Woyczynski (1998), Woyczynski (1998), Biler, Funaki and Woyczynski (2000), Margolius, Subramanian and Woyczynski (2000), for more details on the subject).

For each $n \in \mathbf{N}$, let us introduce independent, symmetric, real-valued standard Brownian motion processes $\{S^i(t), \ \ i = 1, 2, \ldots, n\}$, and let $\delta_\epsilon(x) := (2\pi\epsilon)^{-1/2} \exp\left[-x^2/2\epsilon\right], \epsilon > 0$, be a regularizing kernel. Consider a system of n interacting particles with positions $\{X^i(t)\}_{i=1,\ldots,n} \equiv \{X^{i,n,\epsilon}(t)\}_{i=1,\ldots,n}$, and the corresponding measure-valued process (empirical distribution) $\bar{X}^n(t) \equiv \bar{X}^{n,\epsilon}(t) := \frac{1}{n}\sum_{i=1}^n \delta(X^{i,n,\epsilon}(t))$, with the dynamics provided by the system of regularized singular stochastic differential equations

$$dX^i(t) = dS^i(t) + \frac{1}{n}\sum_{j \neq i} \delta_\epsilon(X^i(t) - X^j(t))\,dt, \quad i = 1, \ldots, n, \tag{68}$$

and the initial conditions $X^i(0) \sim u_0(x)$ (in distribution, thus, $u_0 \in L_1$ here). Then, for each $\epsilon > 0$, the empirical process $\bar{X}^{n,\epsilon}(t) \Longrightarrow u^\epsilon(x,t)\,dx$, in probability, as $n \to \infty$, where $\Rightarrow$ denotes the weak convergence of measures, and the limit density $u^\epsilon \equiv u^\epsilon(x,t),\ t > 0,\ x \in \mathbf{R}$, satisfies the regularized Burgers equation $u_t^\epsilon + (\frac{1}{2}(\delta_\epsilon * u^\epsilon) \cdot u^\epsilon)_x = \Delta u^\epsilon$. with the initial condition $u(0,x) = u_0(x)$. The speed of convergence is controlled (see Bossy and Talay (1996)). Moreover, under some additional technical conditions, for a class of test functions ϕ, $E|\langle \bar{X}^{n,\epsilon(n)}(t) - u(t), \phi\rangle| \longrightarrow 0$, as $n \to \infty, \epsilon(n) \to 0$, where $u(t) = u(x,t)$ is a solution of the nonregularized Burgers equation $u_t + (u^2)_x = \Delta u$ with the initial condition $u(0,x) = u_0(x)$.

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References

[1] A. Benassi and J.P.Fouque, Hydrodynamical limit for the asymmetric simple exclusion process, Ann. of Probability 15(1984), 546-560.

[2] Biler P., Funaki T., Woyczynski W.A., Interacting particle approximation for nonlocal quadratic evolution problems, *Prob. Math. Stat.* **20** (2000), 1–23.

[3] Biler P., Karch G., Woyczynski W.A., Asymptotics of multifractal conservation laws, *Studia Math.* **135** (1999), 231-252.

[4] Biler P., Karch G., Woyczynski W.A., Multifractal and Lévy conservation laws, *Comptes Rendus Acad, Sci (Paris)* (2000), 1–4.

[5] Bossy M., Talay D., Convergence rate for the approximation of the limit law of weakly interacting particles: application to the Burgers equation, *Ann. Appl. Probability* **6** (1996), 818–861.

[6] Calderoni P., Pulvirenti M., Propagation of chaos for Burgers' equation, *Ann. Inst. H. Poincaré - Phys. Th.* **39** (1983), 85–97.

[7] Escobedo M., Zuazua E., Large time behavior for convection-diffusion equations in $\mathbb{R}^N$, *J. Funct. Anal.* **100** (1991), 119–161.

[8] Escobedo M., Vázquez J.L., Zuazua E., Asymptotic behaviour and source-type solutions for a diffusion-convection equation , *Arch. Rational Mech. Anal.* **124** (1993), 43–65.

[9] Funaki T., Woyczynski W.A., Interacting particle approximation for fractal Burgers equation, in *Stochastic Processes and Related Topics: In Memory of Stamatis Cambanis 1943–1995*, I. Karatzas, B.S. Rajput, M.S. Taqqu, Eds., Birkhäuser, Boston (1998), 141–166.

[10] Glynn P.W., Diffusion approximations, in *Handbooks in Operations research and management Science, Vol 2., Stochastic Models*, D.P. Heyman and M.J. Sobel, Eds., North Holland, New York, 1990, pp. 145-198.

[11] Kipnis C., Central limit theorems for infinite series of queues and applications to simple exclusion, *Ann. of Probability* **14** (1986), 397-408.

[12] Liggett T.M., *Interacting Particle Systems*, Springer-Verlag, 1985.

[13] McKean H.P., Propagation of chaos for a class of nonlinear parabolic equations, in *Lecture Series in Differential Equations, VII*, Catholic University, Washington D.C., 1967, pp. 177–194.

[14] Margolius B., A sample path analysis of the $M_t/M_t/c$ queue, *Queuing Systems* **31** (1999), 59–93.

[15] Margolius B., Subramanian N., Woyczynski W.A., A Monte Carlo method for queuing networks which admit nonlinear diffusion approximations, (2000), in preparation.

[16] Smoller J., *Shock Waves and Reaction-Diffusion Equations*, Second Edition, Berlin: Springer-Verlag, 1994 .

[17] Srinivasan R., Queues in series via interacting particle systems, *Math. of Operations Research* **18** (1993), 39-50.

[18] Srinivasan R., Stochastic comparison of density profiles for the road-hog process, *J. Appl. Probability* **28** (1991), 852-863.

[19] Sznitman A.S., Topics in propagation of chaos, 166–251 in *École d'été de St. Flour, XIX - 1989*, Lecture Notes in Math. **1464**, Springer, Berlin, 1991.

[20] Woyczynski W.A., *Burgers–KPZ Turbulence — Göttingen Lectures*, Lecture Notes in Math. 1700, Springer-Verlag, 1998.

[21] Zheng W., Conditional propagation of chaos and a class of quasilinear PDE's, *Ann. of Probability* **23** (1995), 1389–1413.