Chapter 3: Exponential Distribution &Poisson Process

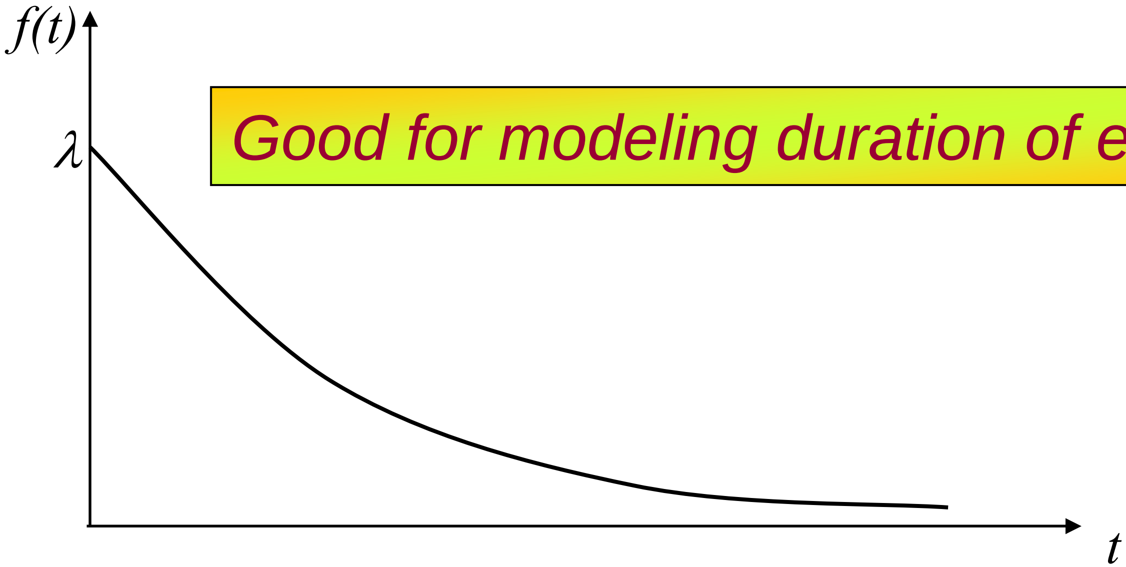
- Definitions
- Properties
- Modeling and Parameters
- Applications

3.1 Exponential Distribution

Definition

A random variable T follows the exponential distribution with rate $\lambda > 0$ if its p.d.f is :

$$f_T(t) = \lambda\, e^{-\lambda t} \qquad \text{for} \quad t > 0$$



Good for modeling duration of events

3.1 Exponential Distribution

Examples

- The life time of an electronic device.
- The time between arrivals of two successive buses.
- The duration time of a car service.
- Time until next earthquake occurs.

Exercise

Give some examples of random variables that could be modeled as exponential random variables from your own experience.

3.1 Exponential Distribution

Properties

1. Distribution Function

$$Pr\{T \le t\} = \int_0^t f(u)du \quad = 1 - e^{-\lambda t}$$

2. Expected Value

$$E[T] = \int_0^\infty tf(t)dt \; = \int_0^\infty t\lambda e^{-\lambda t} dt \; = 1/\lambda$$

Exercise

Prove it?

Note that λ is rate

λ = average number of events in unit of time.

1/λ = average time until an event occurs.

3.1 Exponential Distribution

Properties

3. Variance

$Var[T] = (E[T])^2$

$$Var[T] = E[T^2] - (E[T])^2 = 1/\lambda^2$$

4. Memory less Property

   T is exponentially distributed with rate λ.

$$\Pr\{T > t + h \mid T > t\} = \Pr\{T > h\}$$

3.1 Exponential Distribution

Properties

4. Memory less Property

   T = r. v. lifetime of a battery $\quad\quad T \sim \text{Exp}(\lambda)$

   If the battery worked for t days what is the probability that it will work h additional days?

$$\Pr\{T > t + h \mid T > t\} = \frac{P\{T > t + h \text{ and } T > t\}}{P\{T > t\}} = \frac{P\{T > t + h\}}{P\{T > t\}}$$

$$= \frac{1 - (1 - e^{-\lambda(t+h)})}{1 - (1 - e^{-\lambda t})} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = \Pr\{T > h\}$$

3.1 Exponential Distribution

Example 1

The time required to repair a machine is an exponential random variable with rate $\lambda = 0.5$ downs/hour.

1. What is the probability that a repair time exceeds 2 hours?
2. What is the probability that the repair time will take at least 4 hours given that the repair man has been working on the machine for 3 hours?

Let T : repair time $\Rightarrow f(t) = 0.5\, e^{-0.5t}$ for $t > 0$

1. $P\{T \geq 2\} = e{-}0.5(2) = e{-}1 = 0.36788$
2. $P\{T \geq 4 \mid T \geq 3\} = P\{T \geq 1\} = e^{-0.5(1)} = 0.60653$

3.1 Exponential Distribution

Example 2

Buses arrive to a bus stop according to an exponential distribution with rate $\lambda = 4$ busses/hour.

1. If you arrived at 8:00 am to the bus stop, what is the expected time of the next bus?
2. Assume you asked one of the people waiting for the bus about the arrival time of the last bus and he told you that the last bus left at 7:40 am. What is the expected time of the next bus?

Let T: time between busses $\Rightarrow f(t) = 4\, e^{-4t} \qquad \text{for} \quad t > 0$

3.1 Exponential Distribution

Example 2

1. E[Arrival of next bus | your arrive at 8:00 am]

   = 8:00 + E[T]

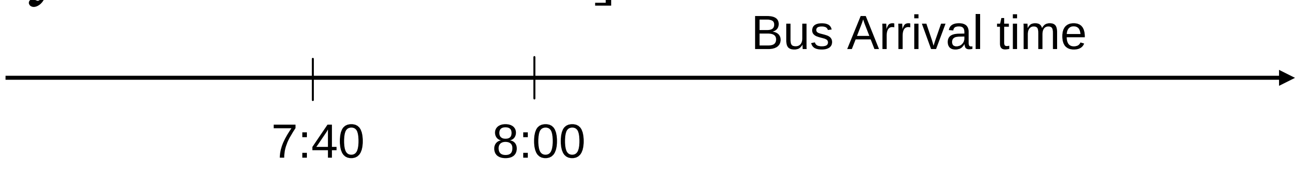
   = 8:00 + 1/λ hour

   = 8:00 + 1/4 hour

   = 8:00 + 15 min

   = 8:15 am

2. E[Time until next bus arrives | last buss arrived at 7:40 and you arrived at 8:00]



   = 8:00 + 1/4 hour = 8:00 + 15 min = 8:15 am

3.1 Exponential Distribution

Exercise

Break downs occur on an old car with rate $\lambda = 5$ break-downs/month. The owner of the car is planning to have a trip on his car for 4 days.

1. What is the probability that he will return home safely on his car.
2. If the car broke down the second day of the trip and the car was fixed, what is the probability that he doesn't return home safely on his car.

3.2 Poisson Process

Definition

A ***random process*** is series of random variables indexed with time.

Examples:

1. Number of arrivals to a system at time t.
2. Number of customers in queue at time t.
3. Number of customers in system at time t.
4. Number of busy servers at time t.

3.2 Poisson Process

Definition

A random process N(t) is a ***Poisson Process*** if :

i. number of events at time $t = 0$ is zero.

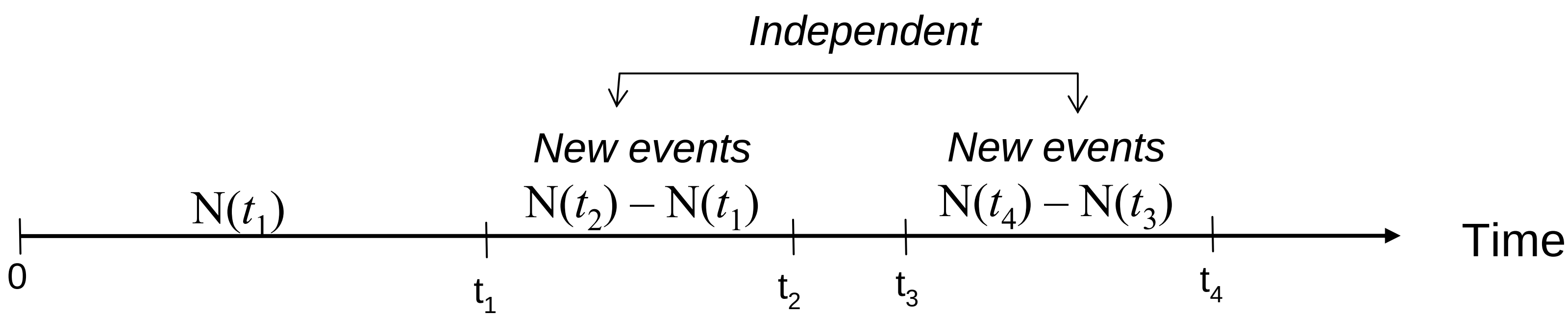
ii. no more than one event can occur in the interval $[0,h]$ for h very small.

iii. number of new events in any disjoint intervals $[t_1, t_2]$ and $[t_3, t_4]$ are independent random variables.

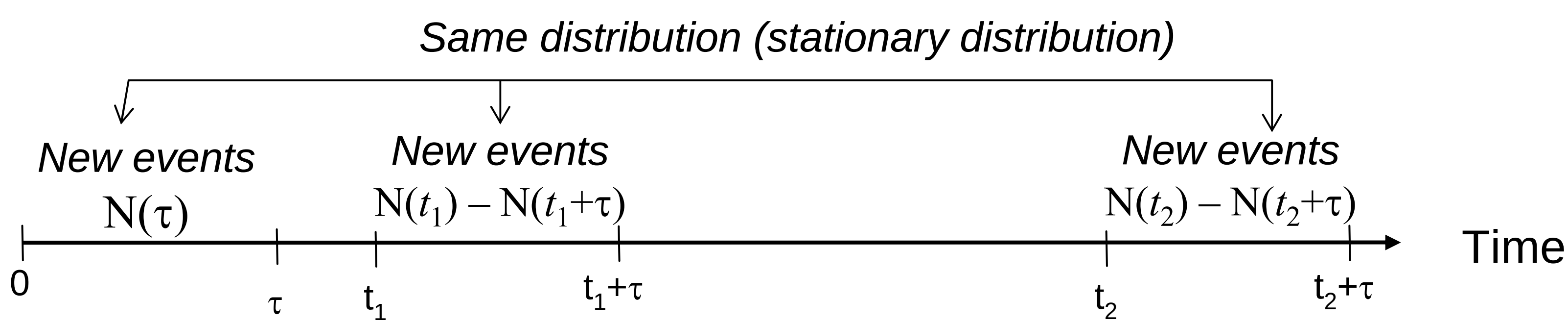
iv. number of new events in any interval of length τ is the same (stationary) for any time.

3.2 Poisson Process

- number of new events in any disjoint intervals $[t_1, t_2]$ and $[t_3, t_4]$ are independent random variables.



- number of new events in any interval of length τ is the same (stationary) for any time.



3.2 Poisson Process

Properties

1. Probability of having k events in the interval $[0\ ,\ t]$ for $t > 0$ is:

$$P\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

2. Probability of having k events in the interval $[s\ ,\ t]$ for $0 \leq s < t$ is:

$$P\{N(t) - N(s) = k\} = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}$$

$$\text{for} \quad k = 0, 1, 2, \ldots$$

3.2 Poisson Process

Properties

3. Expected Value: expected number of events in $[0\ ,\ t]$

$$\mathrm{E}[\mathrm{N}(t)] = \sum_{n=0}^{k} nP\{N(t)=n\} = \lambda t$$

Exercise: expected number of events in $[s\ ,\ t]$??

Var[N] = E [N]

4. Variance: number of events in $[0\ ,\ t]$

$$\mathrm{Var}[N(t)] = \mathrm{E}[N(t)^2] - (\mathrm{E}[N(t)])^2 = \lambda t$$

Exercise: variance of number of events in $[s\ ,\ t]$??

3.2 Poisson Process

Theorem: ***Poisson Process and Exponential Distribution***

If $\{N(t): t \geq 0\}$ is a Poisson process with rate λ then the time between Poisson events is exponential random variable with mean $1/\lambda$.

- if we have to wait an exponential amount of time for an event to occur, then the number of events that already occurred during any interval $[0,t]$ is a Poisson process with mean λt.

3.2 Poisson Process

Example 1:

Customers arrive at a restaurant according to a Poisson distribution at a rate of 20 per hour. The restaurant Opens for business at 11:00 am.

a. The probability of having 20 customers in the restaurant at 11:12 am given that there were 18 customers at 11:07 am.

b. The probability a new customer will arrive between 11:28 and 11:30 am given that the last customer arrived at 11:25 am.

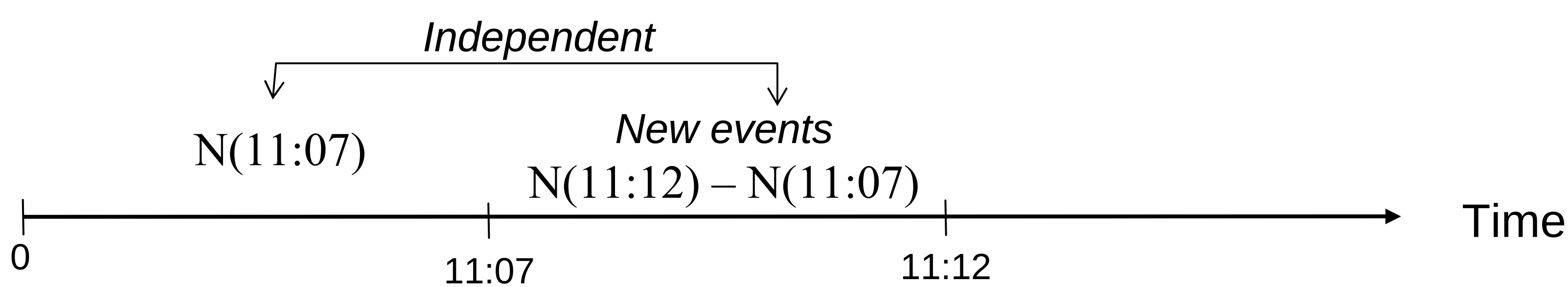
3.2 Poisson Process

Example 1:

$N(t)$ be number of customers arrived during t hours:

$$P\{N(t) = k\} = \frac{(20t)^k}{k!} e^{-\lambda t} \quad , \quad k = 0, 1, 2, \ldots$$

a. P{20 cust. in restaurant at 11:12 am | there were 18 cust. at 11:07am}



$= P\{N(11{:}00–11{:}12) = 20 \mid N(11{:}00–11{:}07) = 18\}$

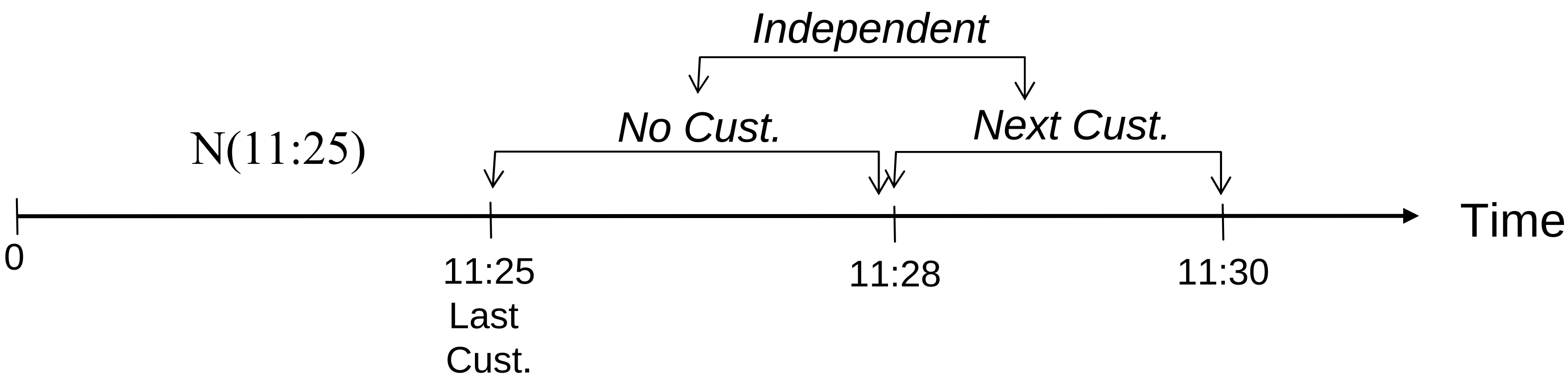
$= P\{N(5 \text{ min}) = 2\}$

$= [20(5/60)]^2 \, e^{-20(5/60)} / 2! = 0.2623$.

3.2 Poisson Process

Example 1:

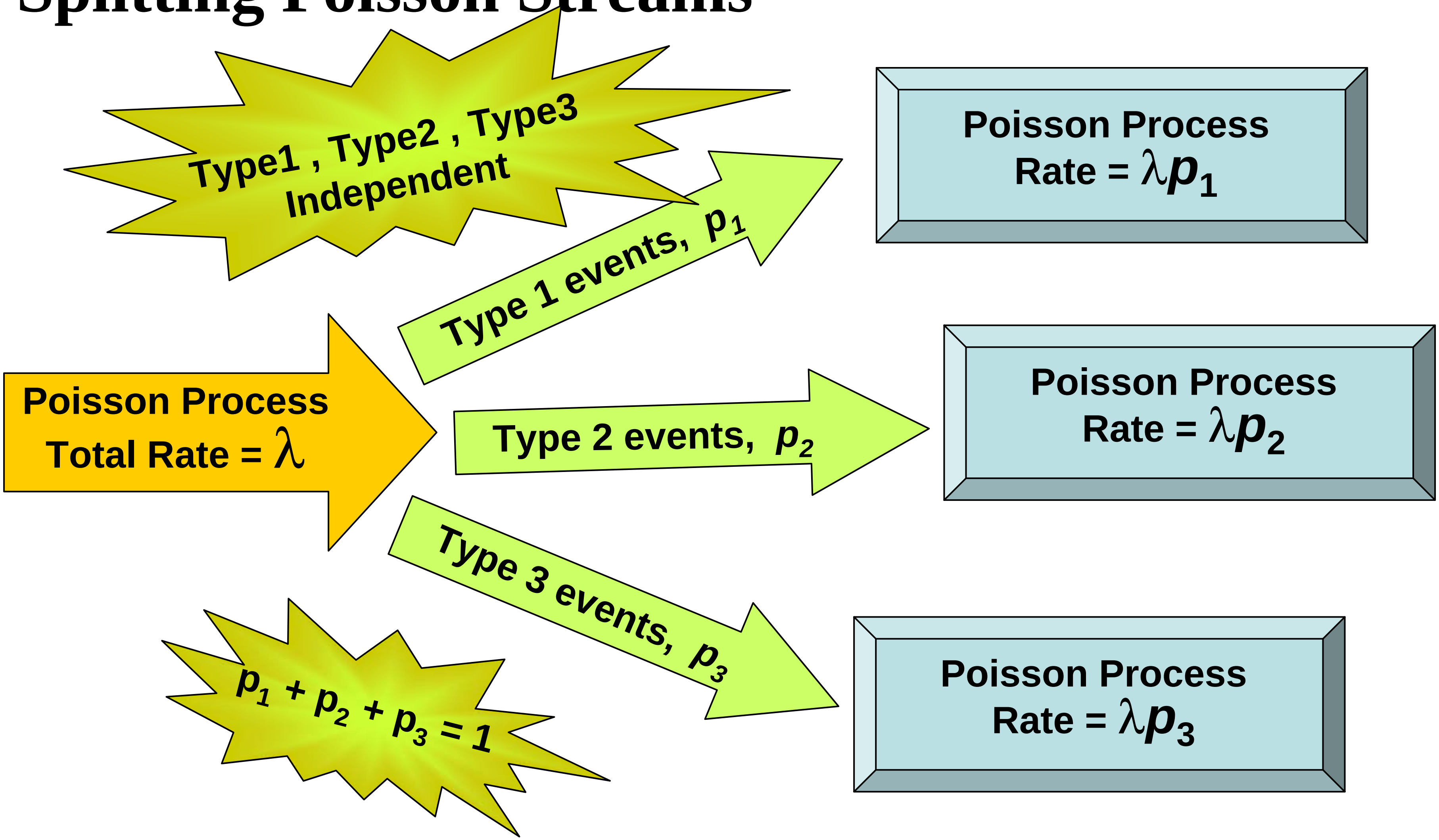
b. $P\{\text{new cust. arrive in } 11{:}28 - 11{:}30\text{am} \mid \text{last cust. arrived at } 11{:}25\text{am}\}$



$$= P\{N(11{:}28\text{–}11{:}30) = 1 \text{ and } N(11{:}25\text{–}11{:}28) = 0\}$$
$$= P\{N(11{:}28\text{–}11{:}30) = 1\} \,.\, P\{N(11{:}25\text{–}11{:}28) = 0\}$$
$$= P\{N(2\text{ min}) = 1\} \,.\, P\{N(3\text{ min}) = 0\}$$
$$= \left([20(2/60)]^1 e^{-20(2/60)}/1! \right) \left([20(3/60)]^0 e^{-20(3/60)} /0! \right)$$
$$= (0.34228)(0.36787) = 0.12591$$

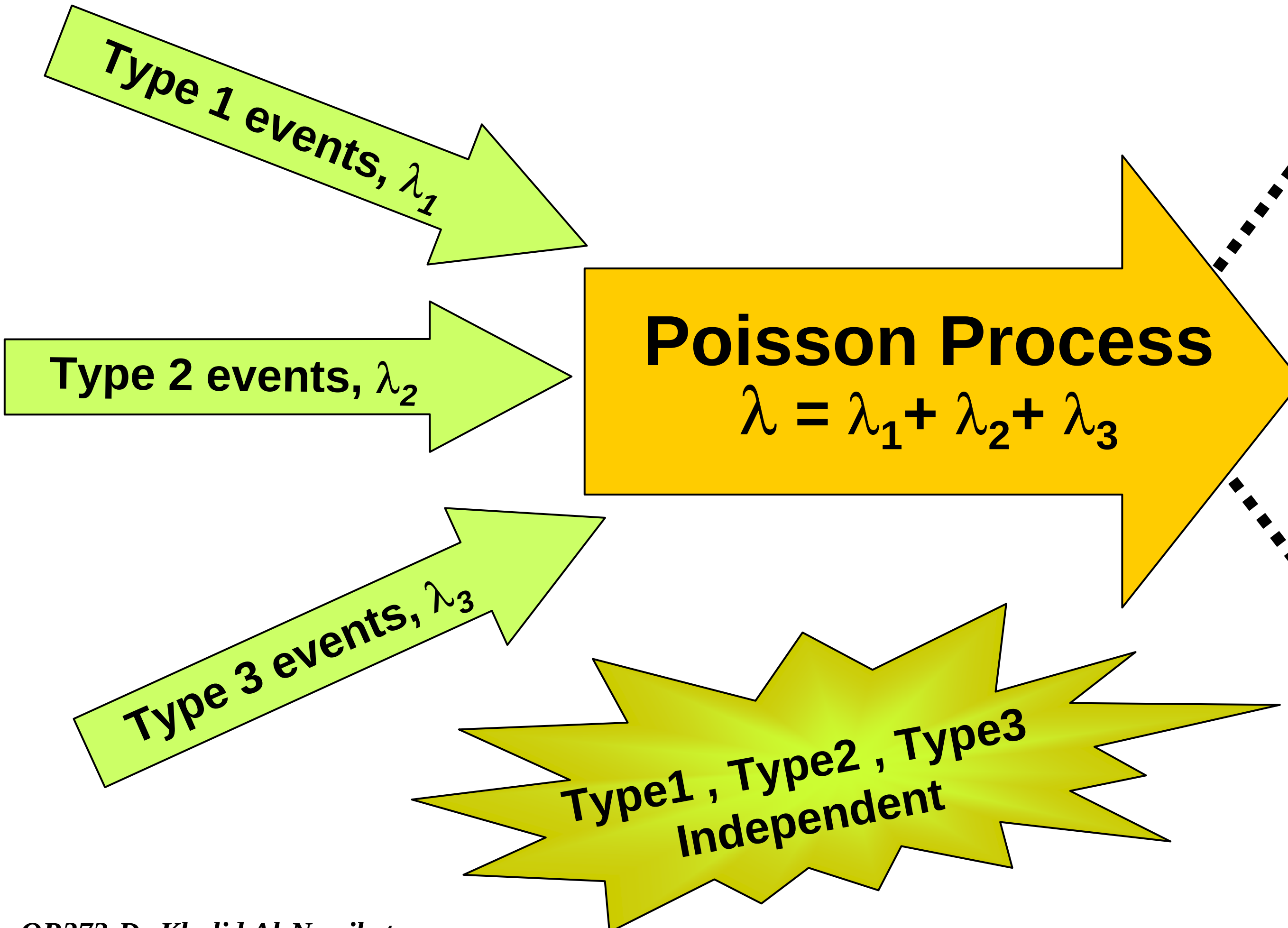
3.2 Poisson Process

Splitting Poisson Streams



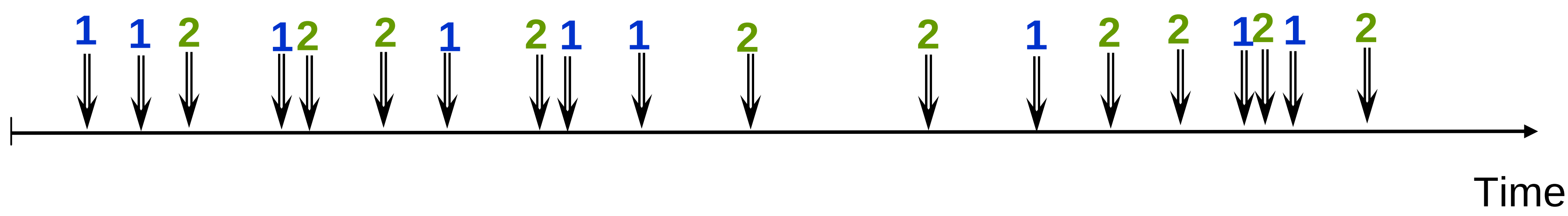
3.2 Poisson Process

Merging Poisson Streams



3.1 Poisson Process

Splitting and Merging Poisson Streams

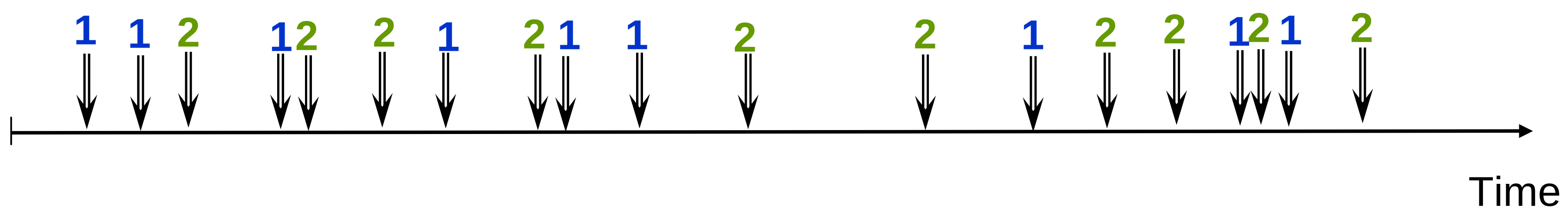


Poisson Event at rate $\lambda = \lambda_1 + \lambda_2$

Poisson Event at rate λ_1

3.2 Poisson Process

Splitting and Merging Poisson Streams



Poisson Event at rate λ

Poisson Event at rate λ_2

3.2 Poisson Process

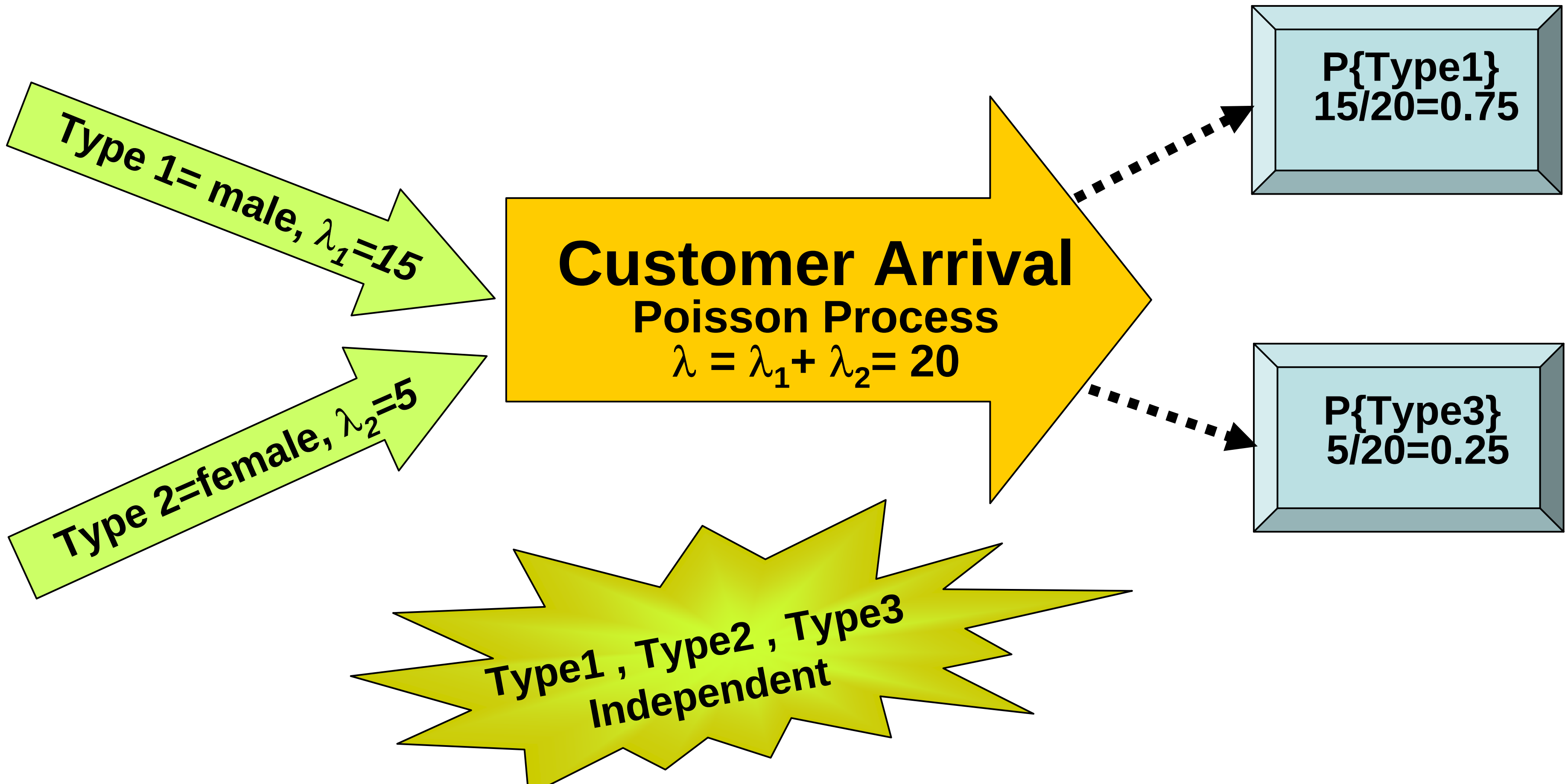
Example 1: *Update*

In the same restaurant it is found that customers could males with rate 15 cust./hr or females at rate 5 cust./hr both types of arrivals are assumed to be Poisson. The restaurant Opens for business at 11:00 am. Find:

a. Probability of having 10 customers in the restaurant at 11:12 am.
b. Probability of having 10 customers in the restaurant at 11:12 am. at least 3 of them are mail.
c. An arrival arrived at 12:15 to the restaurant, what is the probability that he is a female?
d. If no female arrive between 11:30 and 12:30 what is the probability that at most on male arrive between 12:30 and 1:00.
e. If a male arrival arrive to the restaurant at 11:28 what is the number of females he expected to see in restaurant.

3.1 Poisson Process

Example 1: *Update*



3.2 Poisson Process

Example 1: *Update*

$N_m(t)$: no. of male arrives $[0,t]$ $\lambda_m = 15$ cust./hr
$N_f(t)$: no. of female arrives $[0,t]$ $\lambda_f = 5$ cust./hr
Independent Poisson Processes

a. Probability of having 10 customers in the restaurant at 11:12 am.
Pr{Total arrivals = 0 [0,11:12]} = Pr{N(0,11:12)=10, λ=20}

b. Probability of having 10 customers in the restaurant at 11:12 am. at least 3 of them are male.
$\Pr\{\text{ at least 3 male} | \text{ 10 customers arrives}\} = \Pr\{N_m(t)=3, N_f(t)=7\}$
$+ \Pr\{N_m(t)=4, N_f(t)=6\} + \ldots + \Pr\{N_m(t)=10, N_f(t)=0\}$ indep.
$= \Pr\{N_m(t)=3\} \Pr\{N_f(t)=7\} + \Pr\{N_m(t)=4\} \Pr\{N_f(t)=6\} + \ldots$
$+ \Pr\{N_m(t)=10\} \Pr\{N_f(t)=0\}$

3.2 Poisson Process

Example 1: *Update*

$N_m(t)$: no. of male arrives $[0,t]$ $\lambda_m = 15$ cust./hr

$N_f(t)$: no. of female arrives $[0,t]$ $\lambda_f = 5$ cust./hr

Independent Poisson Processes

c. An arrival arrived at 12:15 to the restaurant, what is the probability that he is a female?

d. If no female arrive between 11:30 and 12:30 what is the probability that at most on male arrive between 12:30 and 1:00.

e. If a male arrival arrive to the restaurant at 11:28 what is the number of females he expected to see in restaurant.