Chapter 5: Special Types of Queuing Models

- Some General Queueing Models
- Discouraged Arrivals
- Impatient Arrivals
- Bulk Service and Bulk Arrivals

5.1 General Queueing Models

Very Complex

1. Interarrival time :
   - Could be General distribution
   - Given E[inerarrival time] SD[inerarrival time]
2. Service Time:
   - Could be General distribution
   - Given E[Service time] SD[Service time]
3. System size is infinite or finite

5.2 M/G/1 Queueing Model

Characteristics

1. Interarrival time is exponential with rate λ
   - Arrival process is Poisson Process with rate λ

2. Service time has a any distribution
   - E[Service] = E_s is known $\Rightarrow \mu = 1/E_s$
   - Variance of service time = σ^2

3. Single Server

4. System size is infinite

5. Queue Discipline : FCFS

Notation

M / G / 1 / ∞ / FCFS

5.2 M/G/1 Queueing Model

Steady-State Distribution

State of the system

system is in state **n** if there are n customers in the system

Service Time is not Exponential

Memoryless Property

No Balance Equations

5.2 M/G/1 Queueing Model

Performance Measures

In steady state

$$\text{Given}: \quad \lambda\,,\, E_s\,,\, \sigma^2$$

Know 1 measure
$\Rightarrow$ all measures are known

$$\begin{cases} L_s = L_q + \lambda E_s \\ W_s = W_q + E_s \\ L_s = \lambda W_s \\ L_q = \lambda W_q \end{cases}$$

How to get measures ????

5.2 M/G/1 Queueing Model

Performance Measures

Pollaczek and Kinchin Formula

Given : λ , E_s , σ^2

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda E_s)^2}{2(1-\lambda E_s)}$$

Use relations to find : L_s , W_s and W_q

5.2 M/G/1 Queueing Model

Verify P-K Formula:

Let service time is exponential with rate μ

$\Rightarrow$ $E[S] = E_s = 1/\mu$ $\quad$ $Var[S] = \sigma^2 = 1/\mu^2$

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda E_s)^2}{2(1-\lambda E_s)}$$

5.2 M/G/1 Queueing Model

Verify P-K Formula:

Let service time is exponential with rate μ

$\Rightarrow$ $E[S] = E_s = 1/\mu$ $\quad\quad$ $Var[S] = \sigma^2 = 1/\mu^2$

$$L_q = \frac{\rho^2}{(1-\rho)}$$

$$L_s = L_q + \lambda E_s = \frac{\rho^2}{(1-\rho)} + \lambda(1/\mu) = \frac{\rho^2}{(1-\rho)} + \rho$$

$$= \frac{\rho^2 + \rho(1-\rho)}{(1-\rho)} = \frac{\rho^2 + \rho - \rho^2}{(1-\rho)} = \frac{\rho}{(1-\rho)}$$

5.2 M/G/1 Queueing Model

Example

An average of 15 cars per hour arrive according to a Poisson process at a drive-in to fast food with single window. Assume the service time is uniformly distributed between [2,4] minutes. Answer the following questions in steady-state:

1. What is the average number of cars waiting in line for window?
2. What is the average time a customer spends to get his meal?

Arrival : $\lambda = 15$ cars/hour $= 0.25$ car/min Poisson Process

Service : $S \sim U(2,4) \quad \Rightarrow \quad E_s = (4+2)/2 = 3$ min

$\Rightarrow \quad Var(S) = (4-2)^2/12 = 0.333$

Single window $\Rightarrow$ M/G/1 Queueing system

5.2 M/G/1 Queueing Model

Example

1. Average number of cars waiting in line for window = L_q

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda E_s)^2}{2(1-\lambda E_s)} = \frac{(0.0625)(0.3333) + (0.25(3))^2}{2(1-0.25(3))}$$

$$= \frac{(0.021) + (0.5625)}{0.5} = 1.167$$

2. Average time a customer spends to get his meal = W_s

$$W_s = L_s / \lambda = (L_q + \lambda E_s)/\lambda = (1.167 + 0.333(3))/0.333$$
$$= 6.51 \text{ min}$$

5.3 M/G/s/s Queueing Model

Characteristics

1. Interarrival time is exponential with rate λ
   - Arrival process is Poisson Process with rate λ

2. Service time has a any distribution
   - E[Service] = E_s is known $\Rightarrow \mu = 1/E_s$

3. Multiple Servers = s

4. System size is finite = s

5. Queue Discipline : FCFS

Notation
M / G / s / s / FCFS

5.3 M/G/s/s Queueing Model

Steady-State Distribution

<u>State of the system</u>

system is in state **n** if there are n customers in the system

Steady state distribution exists:

$$P_n = \frac{\dfrac{\rho^n}{n!}}{\displaystyle\sum_{n=0}^{s} \frac{\rho^n}{n!}} \qquad \mathbf{n = 0, 1, 2, \ldots, s}$$

Erlang Distribution

5.3 M/G/s/s Queueing Model

Performance Measures

1. Blocking Probability BP:

$$BP = \Pr\{\text{system is full}\} = P_s = \frac{\frac{\rho^s}{s!}}{\sum_{n=0}^{s} \frac{\rho^n}{n!}}$$ ***Erlang Loss Formula***

2. Effective Arrival Rate λ_e:

$$\lambda_e = \lambda \,.\, (1 - BP) = \lambda \,.\, (1 - P_s)$$

3. Average Customers in System L_s:

$$L_s = \rho(1 - P_s)$$

5.4 G/G/∞ Queueing Model

Characteristics

1. Interarrival time has any distribution
   - E[interarrival time] = E_a is known $\Rightarrow$ $\lambda = 1/E_A$
2. Service time has a any distribution
   - E[Service] = E_s is known $\Rightarrow \mu = 1/E_s$
3. Multiple Servers : $s \rightarrow \infty$
4. System size is infinite $\rightarrow \infty$
5. Queue Discipline : FCFS

Notation

G / G / ∞ / ∞ / FCFS

5.4 G/G/∞ Queueing Model

Steady-State Distribution

State of the system

system is in state **n** if there are n customers in the system

Steady state distribution exists: $\rho = \lambda/\mu$

$$P_n = \frac{\rho^n}{n!} e^{-\rho} \qquad \mathbf{n = 0, 1, 2, \ldots, s}$$

Performance Measure

$$L_s = \rho = \lambda/\mu$$

Poisson Distribution

5.5 Discouraged Arrivals

Characteristics

1. Interarrival time is exponential with rate λ
2. Service time has is exponential with rate μ
3. Multiple Servers : s
4. System size is finite = k
5. Queue Discipline : FCFS

Customers decide not to join the queue as state increases

5.5 Discouraged Arrivals

Steady-State Distribution

State of the system

state **n** = number of customers in the system

n = 0 , 1 , 2 , 3 , …

Probability a customer enters the system when he finds **n** in the system is α^n ; $0 < \alpha < 1$

5.5 Discouraged Arrivals

Steady-State Distribution

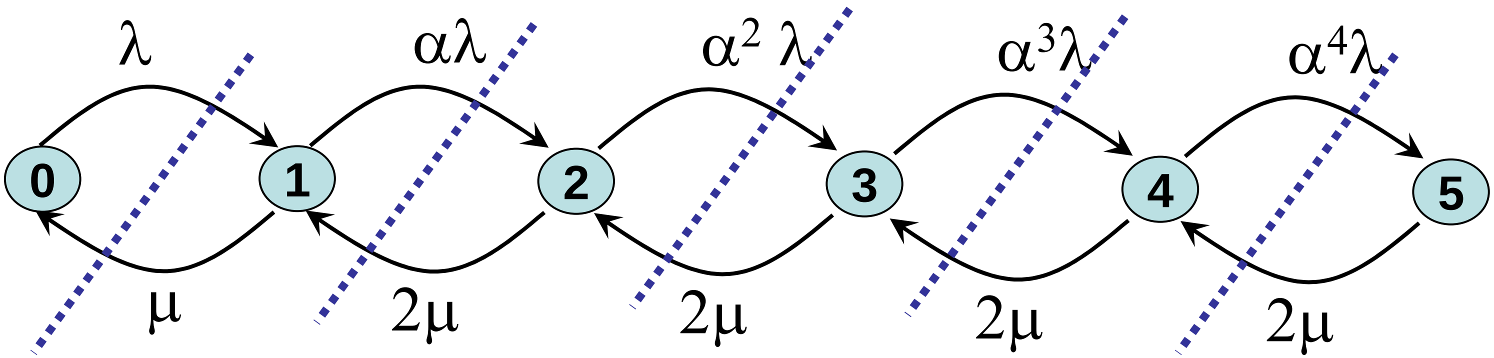
Consider:

- Number of servers = s = 2 identical servers
- System size = k = 5
- Arrival rate = λ
- Service rate = μ
- Entering probability if **n** in the system = α^n $(0<\alpha<1)$

5.5 Discouraged Arrivals

Steady-State Distribution

Balance Equations:



cut-1 $\Rightarrow \quad \lambda P_0 = \mu P_1 \Rightarrow \quad P_1 = (\lambda/\mu)P_0 \quad \Rightarrow \quad P_1 = \rho P_0$

cut-2 $\Rightarrow \alpha\lambda P_1 = 2\mu P_2 \Rightarrow \quad P_2 = (\alpha\lambda/2\mu)P_1 \Rightarrow \quad P_2 = (\alpha\rho^2/2)P_0$

cut-3 $\Rightarrow \alpha^2\lambda P_2 = 2\mu P_3 \Rightarrow \quad P_3 = (\alpha^2\lambda/2\mu)P_2 \Rightarrow \quad P_3 = (\alpha^2\rho^3/4)P_0$

cut-4 $\Rightarrow \alpha^3\lambda P_3 = 2\mu P_4 \Rightarrow \quad P_4 = (\alpha^3\lambda/2\mu)P_3 \Rightarrow \quad P_4 = (\alpha^6\rho^4/8)P_0$

cut-5 $\Rightarrow \alpha^4\lambda P_4 = 2\mu P_5 \Rightarrow \quad P_5 = (\alpha^4\lambda/2\mu)P_4 \Rightarrow \quad P_5 = (\alpha^{10}\rho^5/16)P_0$

$$\sum_{\forall n} P_n = 1$$

5.5 Discouraged Arrivals

Example

Consider an exit on a highway that makes cars move from the highway to the local traffic. At most 5 cars can lineup in the exit. The first car in line takes an exponential time with mean 0.5 min to leave the exit to local traffic. On average, number of cars request the exit 5 cars/min. Any arriving car to the exit find **n** cars present will enter the exit with probability 0.8^n . in the exit Any car finds 5 cars in the exit will not wait. Assume Poisson arrivals. Find:

1. Probability that the exit is empty?
2. Average number of cars enters the exit per hour?
3. Average number of cars in the exit?
4. Average time until a car enters the local traffic?
5. Probability that you cant enter the exit?

5.5 Discouraged Arrivals

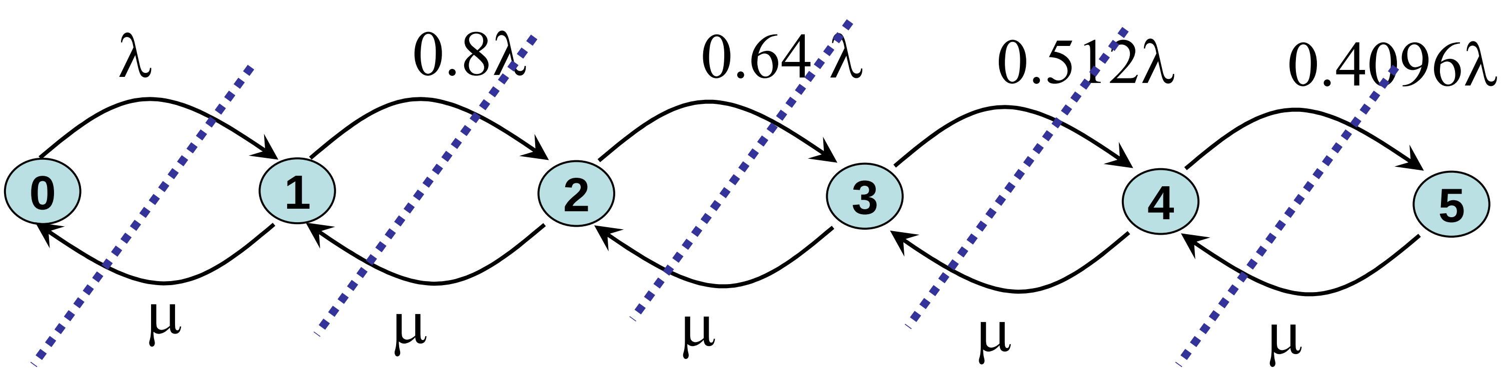
Example

Arrivals : $\lambda = 5$ cars/min Poisson Process

Prob. of Discouraging $\alpha = 0.8$

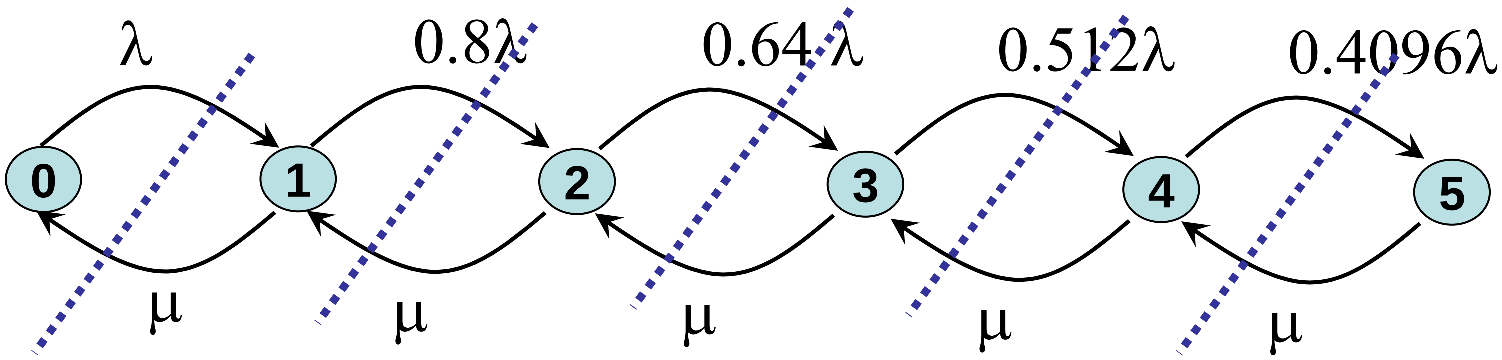
Service: $E[S] = 0.5$ min Exponential

$\Rightarrow \mu = 2$ cars/min Poisson Process



5.5 Discouraged Arrivals

Example



$P_1 = \rho P_0 = (2.5)P_0$

$P_2 = (\alpha\rho^2)P_0 = (5)P_0$

$P_3 = (\alpha^2\rho^3)P_0 = (8)P_0$

$P_4 = (\alpha^6\rho^4)P_0 = (10.24)P_0$

$P_5 = (\alpha^{10}\rho^5)P_0 = (10.485)P_0$

5.5 Discouraged Arrivals

Example

n	0	1	2	3	4	5	Sum
T_n	1	2.5	5	8	10.24	10.485	37.225
P_n	0.0269	0.067	0.134	0.215	0.275	0.282	1.00
nP_n	0	0.067	0.269	0.645	1.100	1.408	3.49

1. Probability that the exit is empty:

$$P_0 \,[1+2.5+5+8+10.24+10.485]^{-1} = 1/37.226 = 0.0269$$

2. Average number of cars enters the exit per hour:

$$\lambda_e(hr) = 60(\lambda P_0 + \alpha\lambda P_1 + \alpha^2\lambda P_2 + \alpha^3\lambda P_3 + \alpha^4\lambda P_4)$$

$$= 60\lambda(P_0 + \alpha\, P_1 + \alpha^2 P_2 + \alpha^3 P_3 + \alpha^4 P_4)$$

$$= 60\ (5)\ (0.389) = 116.7 \ \text{car/hr}$$

5.5 Discouraged Arrivals

Example

3. Average number of cars in the exit:

$L_s = \Sigma\, nP_n = 3.49$ cars

4. Average time until a car enters the local traffic:

$W_s = L_s/\lambda_e = 3.49/116.7 = 0.03$ hr $= 1.8$ min.

5. Probability that you cant enter the exit:

= Pr{exit is full}

$= P_5 = 0.282$

5.6 Impatient Arrivals

Characteristics

1. Interarrival time is exponential with rate λ
2. Service time has is exponential with rate μ
3. Multiple Servers : s
4. System size is finite = k
5. Queue Discipline : FCFS

Customers leave the queue without service after waiting random amount of time

5.6 Impatient Arrivals

Steady-State Distribution

State of the system

state **n** = number of customers in the system

$n = 0, 1, 2, 3, \ldots$

Customer leave the system without service after waiting in queue for exponential time with mean $1/\beta$.

Rate of customers leaving the queue without service = β

5.6 Impatient Arrivals

Steady-State Distribution

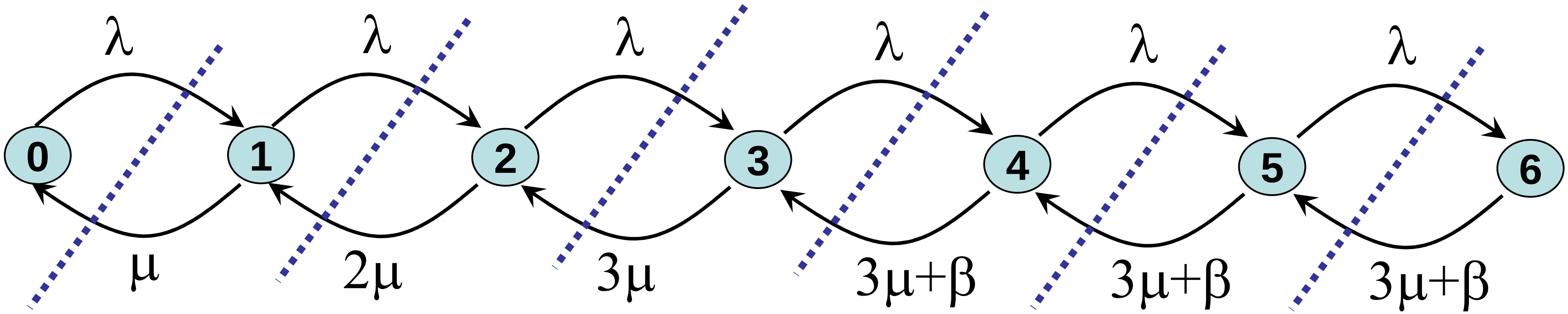
Consider:

- Number of servers = s = 3 identical servers
- System size = k = 6
- Arrival rate = λ
- Service rate = μ
- Rate of impatient customers = β

5.6 Impatient Arrivals

Steady-State Distribution

<u>Balance Equations:</u>



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cut-2 $\Rightarrow \lambda P_1 = 2\mu P_2 \quad \Rightarrow \quad P_2 = (\lambda/2\mu) P_1$

cut-3 $\Rightarrow \lambda P_2 = 3\mu P_3 \quad \Rightarrow \quad P_3 = (\lambda/3\mu) P_2$

cut-4 $\Rightarrow \lambda P_3 = (3\mu+\beta) P_4 \quad \Rightarrow \quad P_4 = \lambda/(3\mu+\beta) P_3$

cut-5 $\Rightarrow \lambda P_4 = (3\mu+\beta) P_5 \quad \Rightarrow \quad P_5 = \lambda/(3\mu+\beta) P_4$

cut-6 $\Rightarrow \lambda P_5 = (3\mu+\beta) P_6 \quad \Rightarrow \quad P_6 = \lambda/(3\mu+\beta) P_5$

$$\sum_{\forall n} P_n = 1$$