4.7 Finite Population Source Model

Characteristics

1. Arrival Process

- R independent Source
- All sources are identical
- Interarrival time is exponential with rate λ for each source
- No arrivals if all sources are in the system.

4.7 Finite Population Source Model

Characteristics

2. Interarrival time is exponential with rate μ
   - Number of services is Poisson Process with rate μ
3. Multiple Server:
   - Number of servers = s
   - Identical, Independent and Parallel servers
   - Random choice of idle servers
4. System size is infinite
5. Queue Discipline : FCFS

Notation

M / M / s / ∞ / FCFS

Finite Population

4.7 Finite Population Source Model

Steady-State Distribution

State of the system

Finite Population = R

system is in state **n** if there are n customers in the system (waiting or serviced)

If system in state n $\Rightarrow$ R−n are expected to arrive

If system in state R $\Rightarrow$ 0 are expected to arrive

Let P_n be probability that there are n customers in the system in the steady-state. $n = 0, 1, 2, 3, \ldots, R$

4.7 Finite Population Source Model

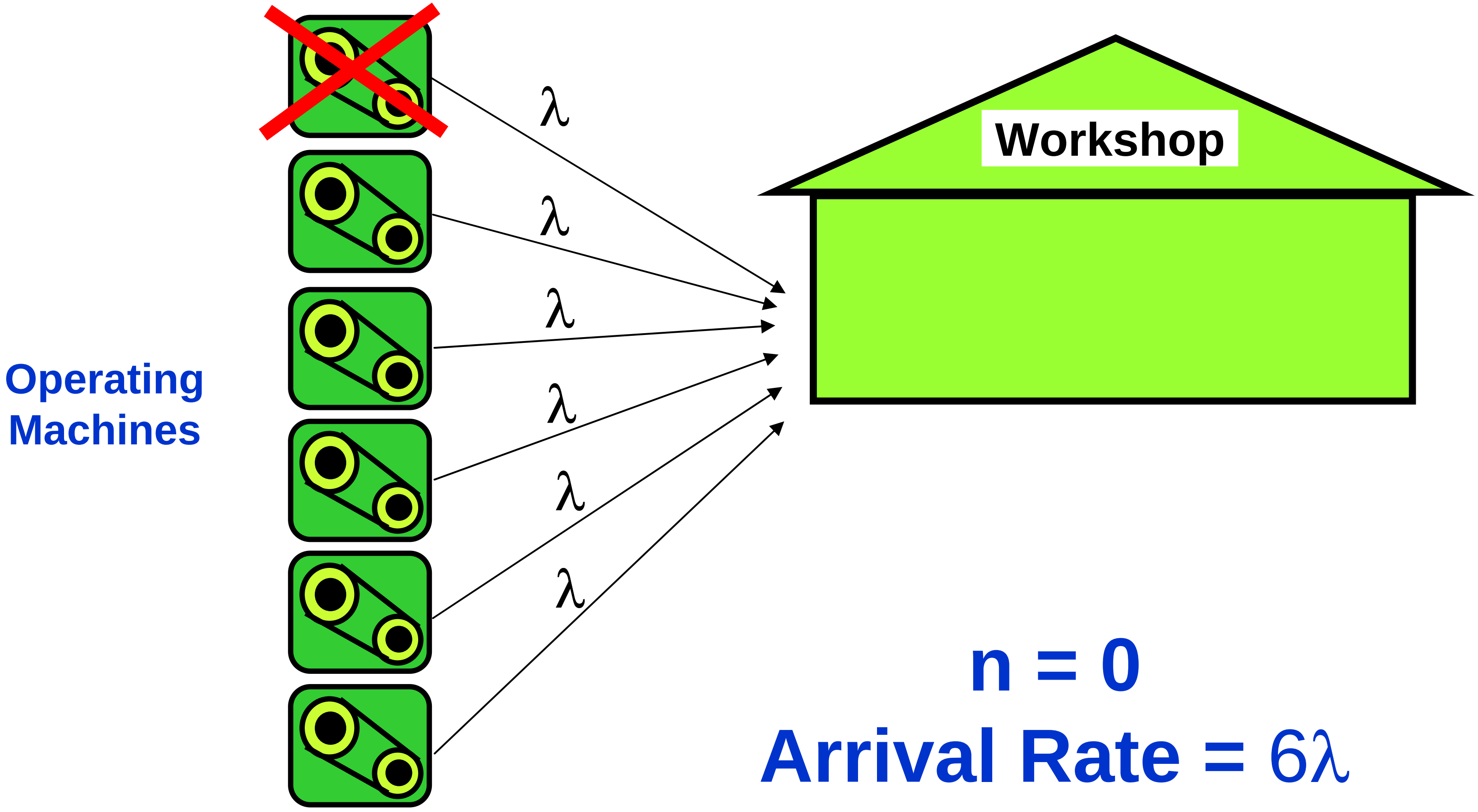
Steady-State Distribution

Consider:

- Facility with 6 identical machines
- Failures on any machine occur at rate λ
- If machine fail is taken to the maintenance
- Each machine needs one worker for maintenance.
- Maintenance workshop has 3 identical workers.
- Each worker works at rate μ

4.7 Finite Population Source Model

Steady-State Distribution

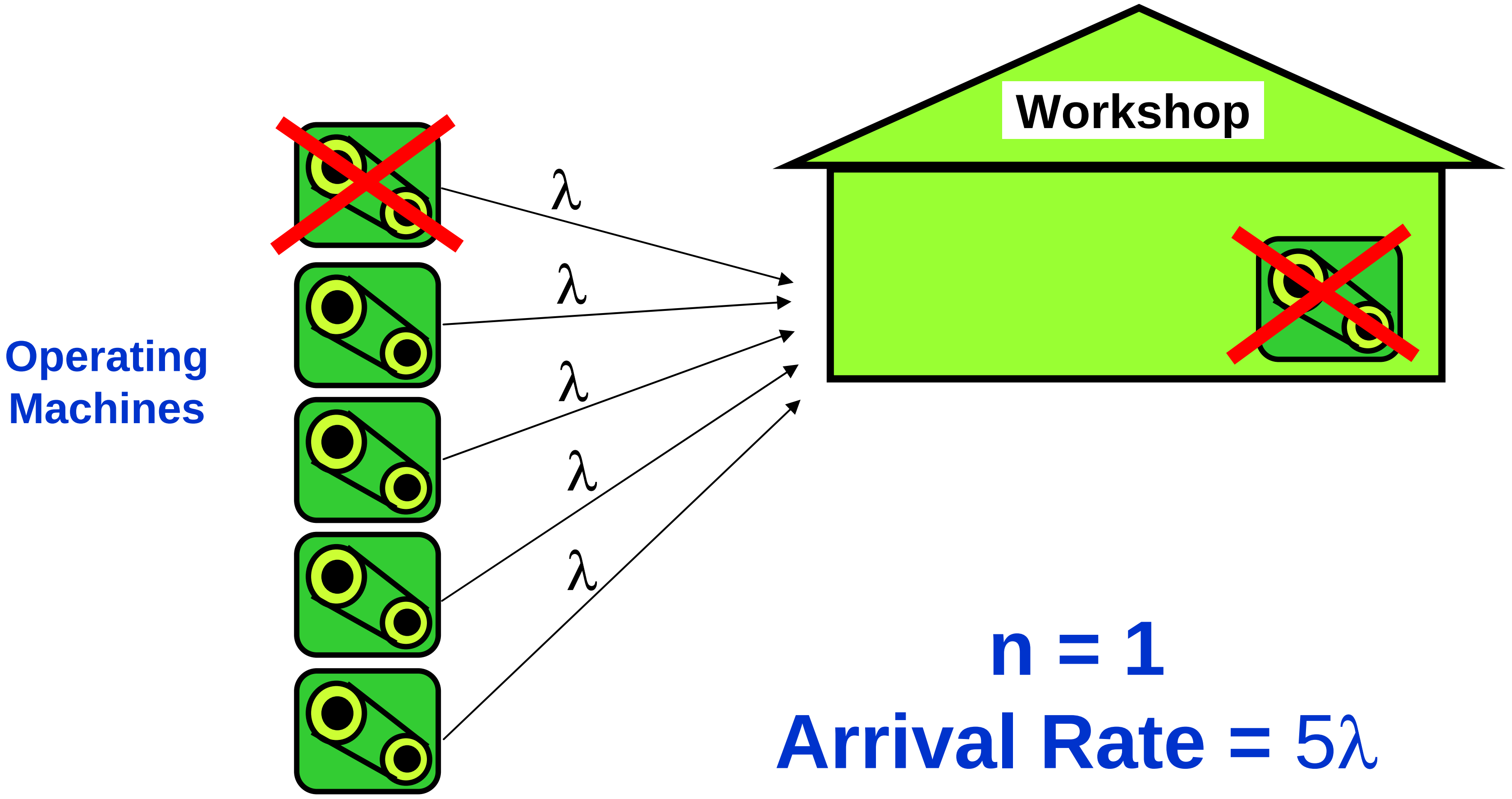


n = 0

Arrival Rate = 6λ

4.7 Finite Population Source Model

Steady-State Distribution

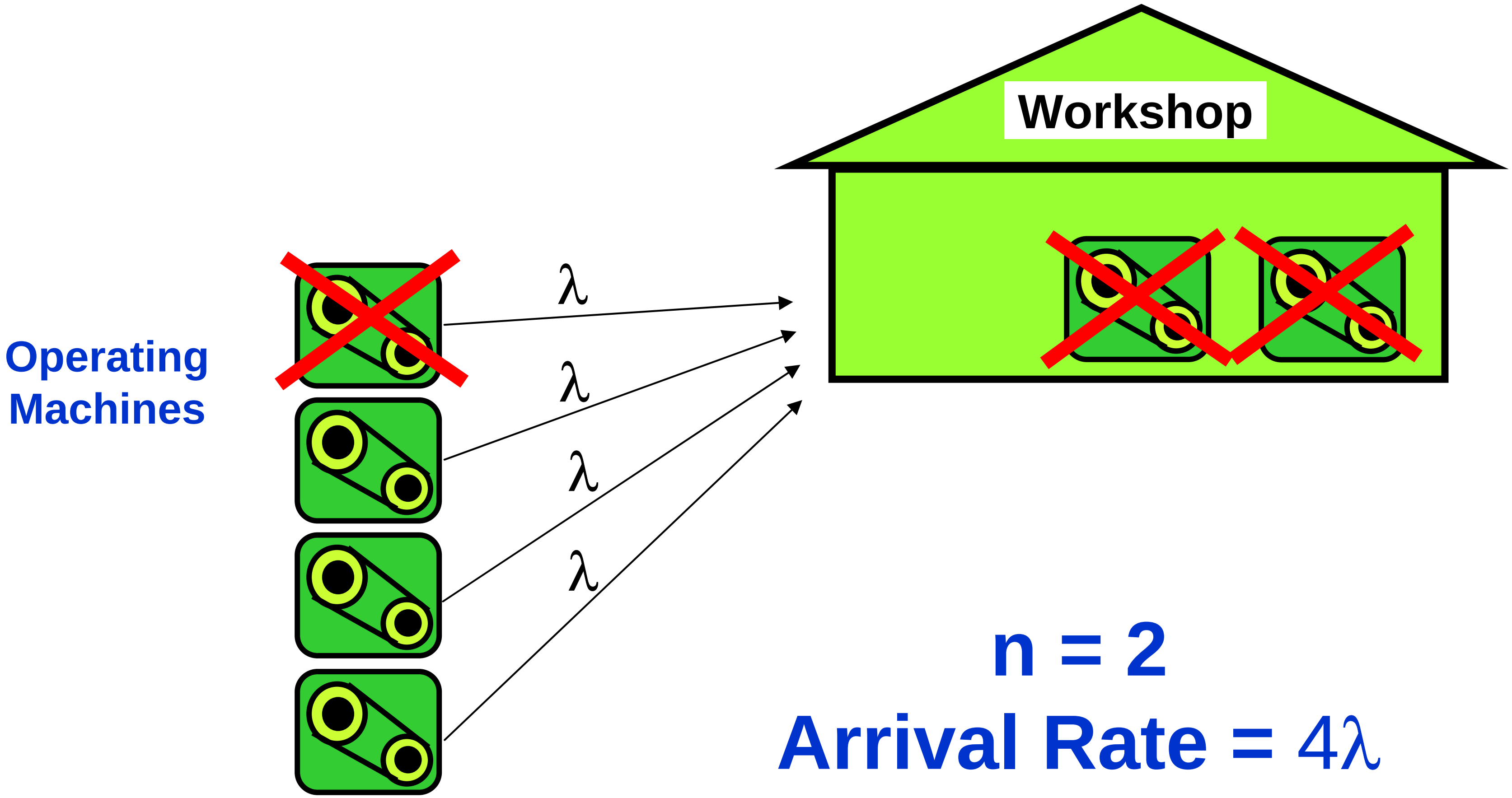


n = 1

Arrival Rate = 5λ

4.7 Finite Population Source Model

Steady-State Distribution



n = 2

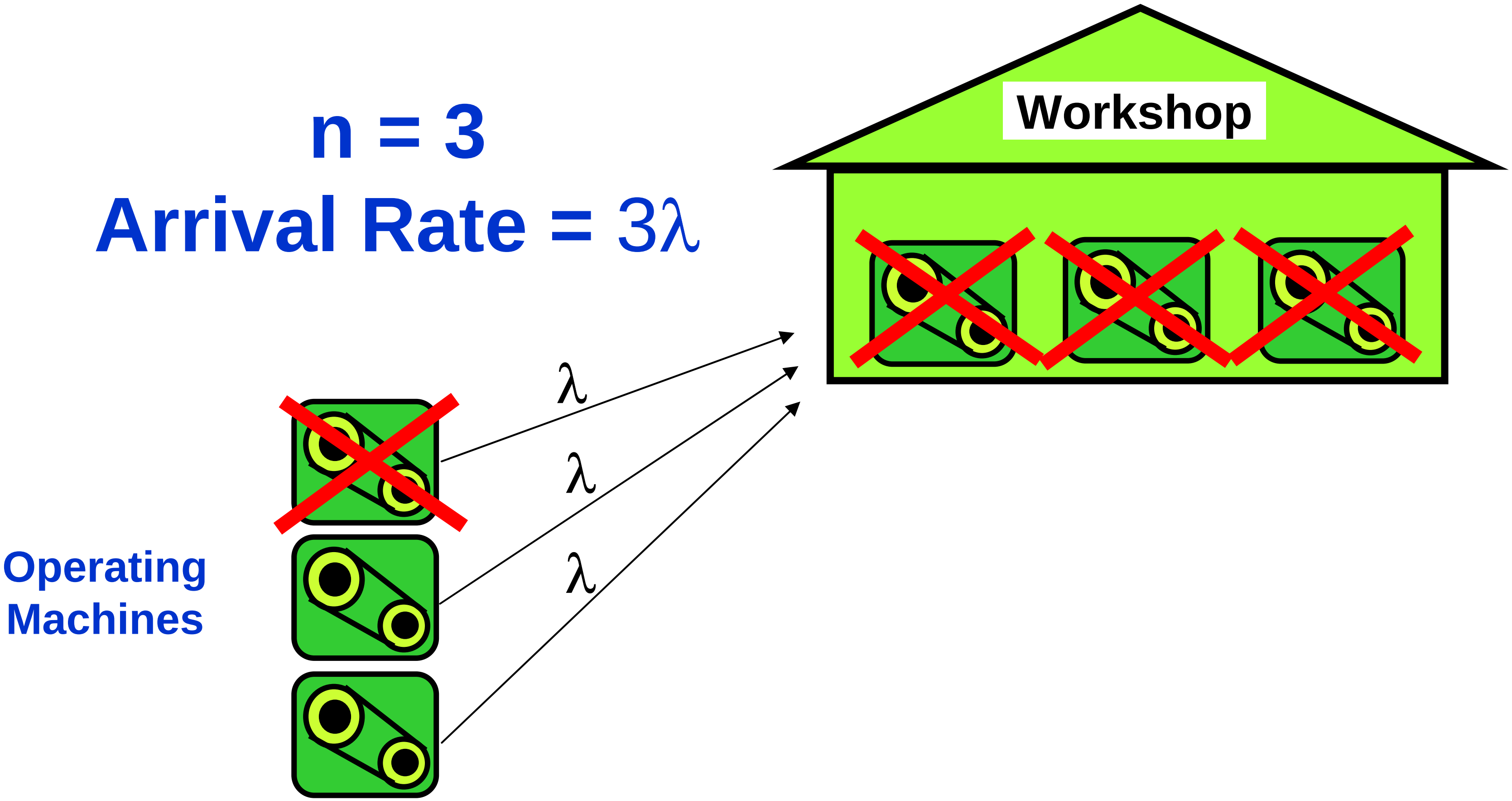
Arrival Rate = 4λ

4.7 Finite Population Source Model

Steady-State Distribution

n = 3

Arrival Rate = 3λ

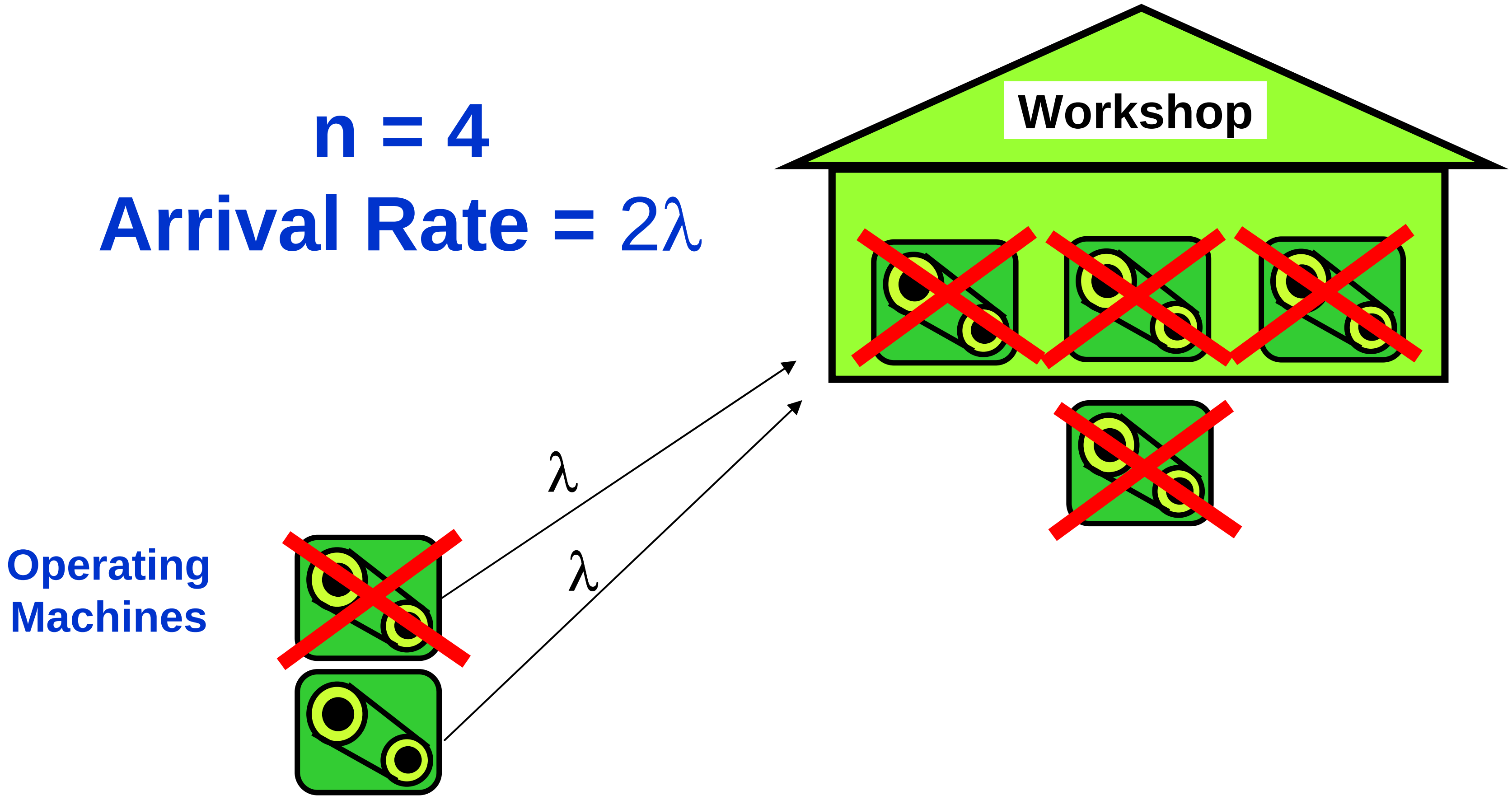


4.7 Finite Population Source Model

Steady-State Distribution

n = 4

Arrival Rate = 2λ

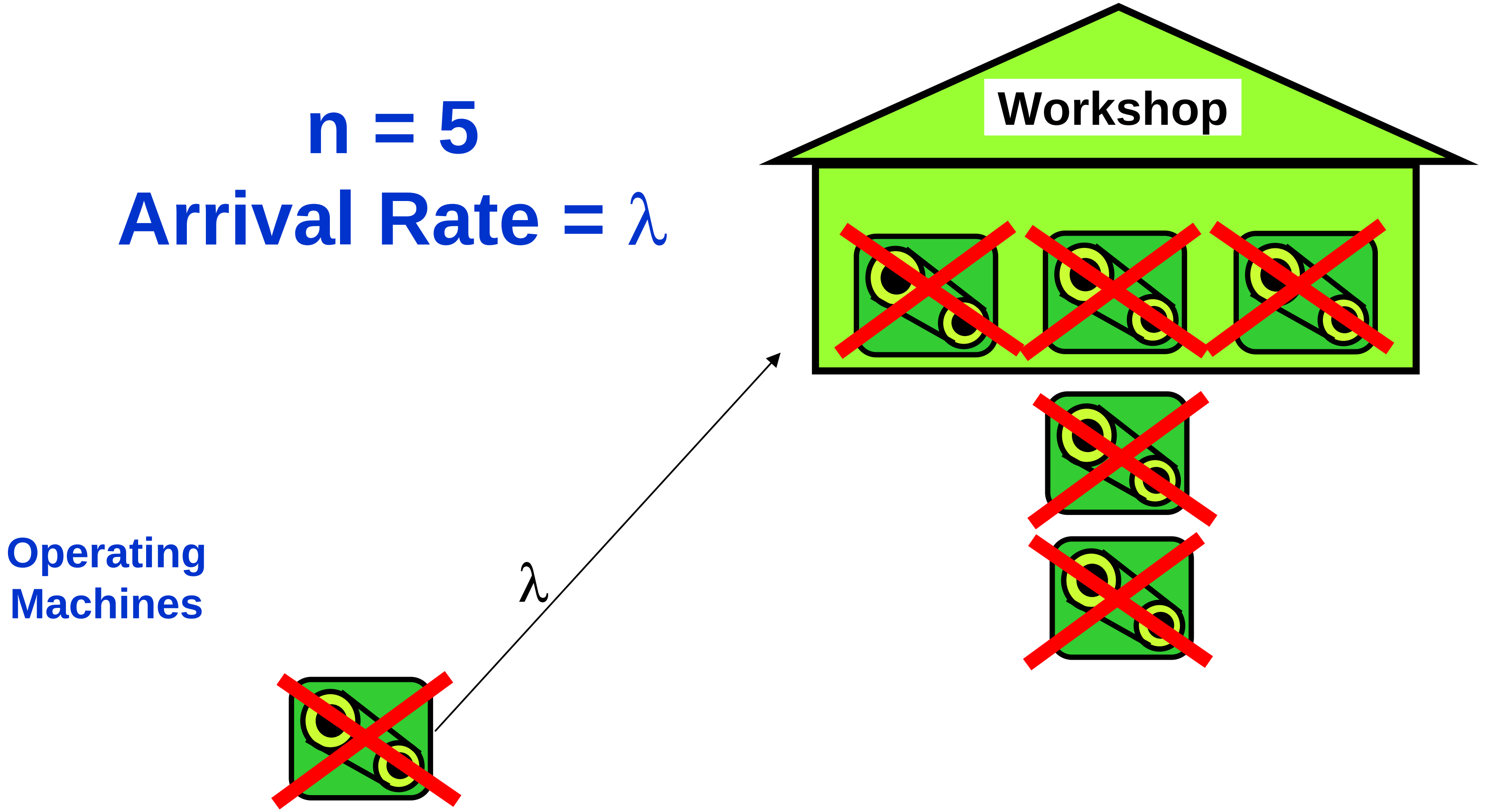


4.7 Finite Population Source Model

Steady-State Distribution

n = 5

Arrival Rate = λ

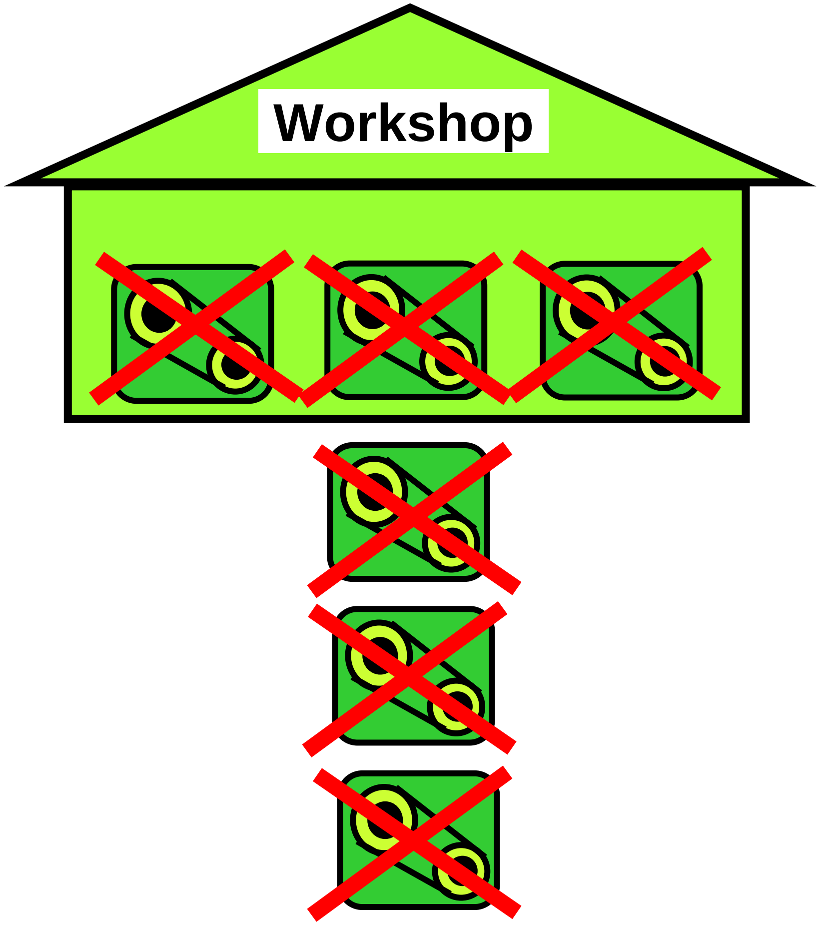


4.7 Finite Population Source Model

Steady-State Distribution

n = 6
Arrival Rate = 0

States of System
n = 0, 1, 2, 3, 4, 5, 6



4.7 Finite Population Source Model

Steady-State Distribution

Rate Diagram:

1. Arrival Rate:

if system in stat n $\Rightarrow$ Arrival rate $= (6 - n)\lambda \quad 0 \le n \le 6$

for number of Machines= R $\Rightarrow$ Arrival rate $= (R - n)\lambda \quad 0 \le n \le R$

2. Service Rate : (Finite Number of servers s = 3)

If system in state $0 < n \le 3 \Rightarrow$ Service rate $= n\mu, \quad 0 \le n \le 3$

If system in state $n > 3 \Rightarrow$ Service rate $= 3\mu, \quad 3 < n \le 6$

for number of servers = s

$\Rightarrow$ Service rate $= n\mu, \quad 0 \le n \le s$

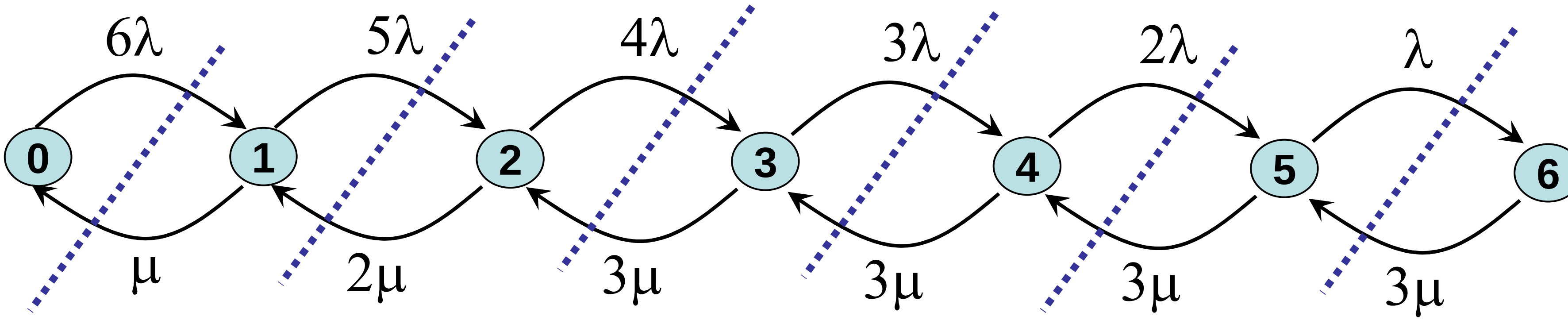
$\Rightarrow$ Service rate $= s\mu, \quad s < n \le R$

4.7 Finite Population Source Model

Steady-State Distribution

$$\begin{bmatrix}\text{Average} \\ \text{Rate } \textbf{out of} \\ \text{State } n\end{bmatrix} = \begin{bmatrix}\text{Average} \\ \text{Rate } \textbf{in to} \\ \text{State } n\end{bmatrix}$$

Balance Equations:



cut-1 ⇒ $6\lambda P_0 = \mu P_1$

cut-2 ⇒ $5\lambda P_1 = 2\mu P_2$

cut-3 ⇒ $4\lambda P_2 = 3\mu P_3$

cut-4 ⇒ $3\lambda P_3 = 3\mu P_4$

cut-5 ⇒ $2\lambda P_4 = 3\mu P_5$

cut-6 ⇒ $\lambda P_5 = 3\mu P_6$

For R-Machines and s-Servers

$0 \le n \le s \Rightarrow \quad (R-n)\lambda P_n = n\mu P_{n+1}$

$s < n < R \Rightarrow \quad (R-n)\lambda P_n = s\mu P_{n+1}$

4.7 Finite Population Source Model

Steady-State Distribution

Solution of Balance Equations:

let $\rho=\lambda/\mu$

$6\lambda P_0 = \mu P_1 \Rightarrow P_1 = (6\lambda/\mu)P_0 = (6)\rho\, P_0$

$5\lambda P_1 = 2\mu P_2 \Rightarrow P_2 = (5\lambda/2\mu)P_1$

$\Rightarrow P_2 = (6.5\lambda^2/2\mu^2)P_0 = (30/2)\rho^2 P_0$

$4\lambda P_2 = 3\mu P_3 \Rightarrow P_3 = (4\lambda/3\mu)P_2$

$\Rightarrow P_3 = (6.5.4\lambda^3/3.2\mu^3)P_0 = (120/6)\rho^3 P_0$

$3\lambda P_3 = 3\mu P_4 \Rightarrow P_4 = (3\lambda/3\mu)P_3$

$\Rightarrow P_4 = (6.5.4.3\lambda^4/3.3.2\mu^4)P_0 = (360/18)\rho^4 P_0$

$2\lambda P_4 = 3\mu P_5 \Rightarrow P_5 = (2\lambda/3\mu)P_4$

$\Rightarrow P_5 = (6.5.4.3.2\lambda^5/3.3.3.2\mu^5)P_0 = (720/54)\rho^5 P_0$

$1\lambda P_5 = 3\mu P_6 \Rightarrow P_6 = (\lambda/3\mu)P_5$

$\Rightarrow P_6 = (6.5.4.3.2\lambda^6/3.3.3.3.2\mu^6)P_0 = (720/162)\rho^6 P_0$

4.7 Finite Population Source Model

Steady-State Distribution

Solution of Balance Equations:

Computing P_0 :

$$\sum_{\forall n} P_n = 1$$

$$P_0 + P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 1$$

$$P_0 + (6)\rho P_0 + (15)\rho^2 P_0 + (20)\rho^3 P_0 + (20)\rho^4 P_0 + (13.33)\rho^5 P_0 + (4.44)\rho^6 P_0 = 1$$

$$P_0[1 + (6)\rho + (15)\rho^2 + (20)\rho^3 + (20)\rho^4 + (13.33)\rho^5 + (4.44)\rho^6] = 1$$

$$P_0 = [1 + (6)\rho + (15)\rho^2 + (20)\rho^3 + (20)\rho^4 + (13.33)\rho^5 + (4.44)\rho^6]^{-1}$$

4.7 Finite Population Source Model

Steady-State Distribution

Solution of Balance Equations:

P_n is a function of P_0
if $P_0 = 0 \Rightarrow$ system is infinite
system is finite if an only if $P_0 > 0$

Computing P_0 :

$$P_0 = \sum_{n=0}^{R} P_n$$

$$\frac{\rho}{s} = \frac{\lambda}{s\mu} < 1$$

NO

finite sum $\Rightarrow$ finite value

$P_0 > 0$ and finite always for any λ , μ and s

No Steady-State Condition on λ , μ , and s

4.7 Finite Population Source Model

Performance Measures

In steady state

$$\lambda_e \, , \, \mu \, , \, P_0$$

$$L_B = E[\text{busy servers}] = E[\#\text{Cust. in service}]$$

$$L_s = L_q + L_B$$

$$W_s = W_q + (1/\mu)$$

$$L_s = \lambda_e W_s$$

$$L_q = \lambda_e W_q$$

$$L_B = \lambda_e W_B$$

System is in Steady Stead

Know 4 measures $\Rightarrow$ all measures are known

4.7 Finite Population Source Model

Performance Measures

1. Effective Arrival Rate λ_e:

λ_e = E[arrival rate]

$$= \sum_{n=0}^{R} (R-n)\lambda\, P_n = \lambda \sum_{n=0}^{R} (R-n)\, P_n$$

$$= \lambda\left(\sum_{n=0}^{R} RP_n - \sum_{n=0}^{R} n\, P_n \right) = \lambda\,(R - L_s)$$

4.7 Finite Population Source Model

Performance Measures

2. Average Customers in System L_s:

$$L_s = \sum_{n=0}^{R} n \cdot P_n$$

3. Average Busy servers L_B:

L_B = E[busy servers] = E[#Cust. in service]

$L_B = 0.P_0 + 1.P_1 + 2.P_2 + \ldots + s\,(\Pr\{ n \geq s \})$

$L_B = 0.P_0 + 1.P_1 + 2.P_2 + \ldots + s\,(1 - \Pr\{ n < s \})$

4.7 Finite Population Source Model

Performance Measures

4. Average Customers in Queue L_q:

$$L_q = L_s - L_B$$

$$L_q = 0.(P_0 + P_1 + P_2 + \ldots + P_s) + 1.P_{s+1} + 2.P_{s+2} + \ldots + 2.P_R$$

$$= \sum_{n=s}^{R} (n-s) \cdot P_n$$

5. Utilization of the System U:

$$U = \Pr\{\, n > 0 \,\} = P_1 + P_2 + P_3 + \ldots + P_R = 1 - P_0$$

6. Utilization of the Service SU:

$$SU = \Pr\{\text{ all servers busy }\} = \Pr\{\, n \geq s \,\}$$

$$SU = 1 - (P_0 + P_1 + P_2 + \ldots + P_{s-1})$$

4.7 Finite Population Source Model

Performance Measures

7. Average Time Spent in System W_s:

$$L_s = \lambda_e . W_s \quad \Leftrightarrow \quad \boxed{W_s = \frac{L_s}{\lambda_e}}$$

8. Average Waiting time in Queue W_q:

$$L_q = \lambda_e . W_q \quad \Leftrightarrow \quad \boxed{W_q = \frac{L_q}{\lambda_e}}$$

4.7 Finite Population Source Model

Example:

A factory has 6 identical machines for production. Failures occur on each machine at a rate of 2 failures per day according to a Poisson process. The maintenance department has 3 workers all have the same experience. Once a machine failed, one of the workers is called to repair it. If all workers are busy the machine is put in awaiting list for repair when workers available. The repair time is exponentially distributed with mean 3 hours. Assume factory works 9 hrs a day.

1. What is the probability that all worker are idle?
2. On average how many machine repaired in one day?
3. On average how many machine waiting for repaired ?
4. What is the probability that all servers are busy?
5. What is the expected time until a failed machine to restart operation?

4.7 Finite Population Source Model

Example:

Arrivals: $\lambda = 2$ failures/day Poisson

$= 0.222$ failures/hr

Service: $E[S] = 3$hr Exponential

$\Rightarrow \mu = 1/E[S] = 1/3$ failures/hr

$\rho = \lambda/\mu = 0.222/0.333 = 0.667$

Number of Servers: $s = 3$

Population size $= R = 6$

$n = 0,1,2,3,4,5,6$

$\Rightarrow$ M/M/3 Finite Population R=6

4.7 Finite Population Source Model

Example:

Use rate diagram:

$$P_1 = (6\lambda/\mu)P_0 \qquad = (6)\rho\, P_0$$
$$P_2 = (6.5\lambda^2/2\mu^2)P_0 = (30/2)\rho^2 P_0$$
$$P_3 = (6.5.4\lambda^3/3.2\mu^3)P_0 = (120/6)\rho^3 P_0$$
$$P_4 = (6.5.4.3\lambda^4/3.3.2\mu^4)P_0 = (360/18)\rho^4 P_0$$
$$P_5 = (6.5.4.3.2\lambda^5/3.3.3.2\mu^5)P_0 = (720/54)\rho^5 P_0$$
$$P_6 = (6.5.4.3.2\lambda^6/3.3.3.3.2\mu^6)P_0 = (720/162)\rho^6 P_0$$

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**Sum.**
T_n	1	4	2.667	1.778	1.185	0.79	0.527	23.711
P_n	0.0837	0.335	.223	0.149	0.099	0.066	0.044	1.00
nP_n	0	0.335	0.446	0.446	0.397	0.331	0.265	2.22

4.7 Finite Population Source Model

Example:

1. Pr{worker are idle}

   = P{all machines working} = $P_0 = 0.0837$

2. Average machines repaired in one day = λ_e

   $= \lambda\,(R - L_s) = 2\,(6 - 2.22) = 7.56$ machines/day

3. Average machine waiting for repaired = L_q

$$L_q = \sum_{n=s}^{R} (n - s) \cdot P_n = 1.P_4 + 2.P_5 + 3.P_6 = 0.364 \text{ machine}$$

4.7 Finite Population Source Model

Example:

4. Pr{all servers are busy}

$= P_3 + P_4 + P_5 + P_6 = 0.358$

5. Expected time until a failed machine return to operation

$= W_s = L_s / \lambda_e = 2.22/7.56$

$= 0.293$ hrs