4.4 M/M/s/k Queueing Model

Characteristics
1. Interarrival time is exponential with rate $\lambda$
   - Arrival process is Poisson Process with rate $\lambda$
2. Interarrival time is exponential with rate $\mu$
   - Number of services is Poisson Process with rate $\mu$
3. Multiple Server:
   - Number of servers = $s$
   - Independent servers
   - Parallel service channels
   - Identical servers
   - Random choice of idle servers
4. System size is finite: Total number in system $\leq k$
5. Queue Discipline: FCFS

Notation
\[ M / M / s / k / FCFS \]
4.4 M/M/s/k Queueing Model

Steady-State Distribution

State of the system

system is in state \( n \) if there are \( n \) customers in the
system (waiting or serviced)

Let \( P_n \) be probability that there are \( n \) customers in the
system in the steady-state. \( n = 0 , 1 , 2 , 3 , \ldots , k \)
4.4 M/M/s/k Queueing Model

Steady-State Distribution

Rate Diagram:
Consider an M/M/3/k system

- Queue is closed
  - system size = k

Rate Diagram:

- System in state \( n = 1 \) ⇒ 1 server busy ⇒ total service speed = \( \mu \)
- System in state \( n = 2 \) ⇒ 2 servers busy ⇒ total service speed = \( 2\mu \)
- System in state \( n = 3 \) ⇒ 3 servers busy ⇒ total service speed = \( 3\mu \)
- System in state \( n = 4 \) ⇒ 3 servers busy ⇒ total service speed = \( 3\mu \)
- …
- System in state \( n > 3 \) ⇒ 3 servers busy ⇒ total service speed = \( 3\mu \)
4.4 M/M/s/k Queueing Model

Steady-State Distribution

Rate Diagram:
1. Arrival rate = λ for all states of the system
2. Service rate = n\mu if n ≤ s
   Service rate = s\mu if s < n ≤ k

```
\[ \begin{array}{ccc}
0 & 1 & 2 \\
\lambda & \mu & 2\mu \\
\end{array} \quad \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\lambda & (s-1)\mu & s\mu \\
\end{array} \quad \begin{array}{ccc}
s & s+1 \\
\lambda & s\mu & s\mu \\
\end{array} \quad \begin{array}{ccc}
k & k-1 \\
\lambda & s\mu & s\mu \\
\end{array} \quad \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\lambda & \mu & 2\mu \\
\end{array} \quad \begin{array}{ccc}
0 & 1 & 2 \\
\end{array} \]
```
4.4 M/M/s/k Queueing Model

Steady-State Distribution

Balance Equations:

\[
\begin{align*}
\text{n = 0 } & \Rightarrow \lambda P_0 = \mu P_1 \\
\text{n = 1 } & \Rightarrow \lambda P_1 + \mu P_1 = \lambda P_0 + 2\mu P_2 \quad \Leftrightarrow (\lambda + \mu)P_1 = \lambda P_0 + 2\mu P_2 \\
\text{n = 2 } & \Rightarrow \lambda P_2 + 2\mu P_2 = \lambda P_1 + 3\mu P_3 \quad \Leftrightarrow (\lambda + 2\mu)P_2 = \lambda P_1 + 3\mu P_3 \\
\text{n = 3 } & \Rightarrow \lambda P_3 + 3\mu P_3 = \lambda P_2 + 4\mu P_4 \quad \Leftrightarrow (\lambda + 3\mu)P_3 = \lambda P_2 + 4\mu P_4 \\
\text{\ldots \ldots \ldots} \\
\text{n = s } & \Rightarrow \lambda P_s + s\mu P_s = \lambda P_{s-1} + s\mu P_{s+1} \Leftrightarrow (\lambda + s\mu)P_s = \lambda P_{s-1} + s\mu P_{s+1}
\end{align*}
\]
4.4 M/M/s/k Queueing Model

Steady-State Distribution

Balance Equations:

\[
\begin{align*}
\lambda P_s + s\mu P_s &= \lambda P_{s-1} + s\mu P_{s+1} \\
\Rightarrow (\lambda + s\mu)P_s &= \lambda P_{s-1} + s\mu P_{s+1} \\
\lambda P_{s+1} + s\mu P_{s+1} &= \lambda P_s + s\mu P_{s+2} \\
\Rightarrow (\lambda + s\mu)P_{s+1} &= \lambda P_s + s\mu P_{s+2} \\
\lambda P_{k-1} &= s\mu P_k \\
\Rightarrow s\mu P_k &= \lambda P_{k-1}
\end{align*}
\]
4.4 M/M/s/k Queueing Model

Steady-State Distribution
Solution of Balance Equations:

Eq-1: \( \lambda P_0 = \mu P_1 \)

Eq-2: \( (\lambda + \mu)P_1 = \lambda P_0 + 2\mu P_2 \)

Eq-3: \( (\lambda + 2\mu)P_2 = \lambda P_1 + 3\mu P_3 \)

\[ \cdots \]

Eq-n: \( (\lambda + n\mu)P_n = \lambda P_{n-1} + (n+1)\mu P_{n+1} \) \( n < s \)

\[ \cdots \]

Eq-n: \( (\lambda + s\mu)P_n = \lambda P_{n-1} + s\mu P_{n+1} \) \( n \geq s \)

\[ \cdots \]

Eq-k: \( s\mu P_k = \lambda P_{k-1} \)
4.4 M/M/s/k Queueing Model

Steady-State Distribution
Solution of Balance Equations:

Eq-1 ⇔ \( \lambda P_0 = \mu P_1 \)  
Eq-2 ⇔ \( \lambda P_1 = 2\mu P_2 \)  
Eq-3 ⇔ \( \lambda P_2 = 3\mu P_3 \)  

\[ \begin{array}{l}
\vdots \\
\text{Eq-n} \quad \lambda P_{n-1} = n\mu P_n, \ n < s \\
\vdots \\
\text{Eq-k} \quad \lambda P_{k-1} = s\mu P_k \\
\end{array} \]

⇔ \( P_1 = (\lambda/\mu) P_0 \)  
⇔ \( P_2 = (\lambda/2\mu) P_3 \)  
⇔ \( P_3 = (\lambda/3\mu) P_2 \)  

\[ \begin{array}{l}
\vdots \\
\text{Eq-n} \quad P_n = (\lambda/n\mu) P_{n-1} \\
\vdots \\
\text{Eq-k} \quad P_k = (\lambda/s\mu) P_{k-1} \\
\end{array} \]
4.4 M/M/s/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:
Make all equations functions of $P_0$ only:

\[
\begin{align*}
\text{Eq-1} & \iff P_1 = \left(\frac{\lambda}{\mu}\right) P_0 \\
\text{Eq-2} & \iff P_2 = \left(\frac{\lambda}{2\mu}\right) P_1 \\
\text{Eq-3} & \iff P_3 = \left(\frac{\lambda}{3\mu}\right) P_2 \\
\text{Eq-n} & \iff P_n = \left(\frac{\lambda}{n\mu}\right) P_{n-1}, n < s \\
\text{Eq-n} & \iff P_n = \left(\frac{\lambda}{s\mu}\right) P_{n-1}, n \geq s \\
\text{Eq-k} & \iff P_k = \left(\frac{\lambda}{s\mu}\right) P_{k-1}
\end{align*}
\]
4.4 M/M/s/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:
Make all equations functions of $P_0$ only:

Let $\rho = \frac{\lambda}{\mu}$

Eq-1 $\iff$ $P_1 = \rho P_0$

Eq-2 $\iff$ $P_2 = \left(\frac{\rho^2}{2}\right)P_0$

Eq-3 $\iff$ $P_3 = \left(\frac{\rho^3}{6}\right)P_0$

\[\cdots\cdots\]

Eq-n $\iff$ $P_n = \left(\frac{\rho^n}{n!}\right)P_0 \quad n < s$

\[\cdots\cdots\]

Eq-n $\iff$ $P_n = \left[\frac{\rho^n}{(s! \ s^{n-s})}\right]P_0 \quad n \geq s$

\[\cdots\cdots\]

Eq-k $\iff$ $P_k = \left[\frac{\rho^k}{(s! \ s^{k-s})}\right]P_0$
4.4 M/M/s/k Queueing Model

Steady-State Distribution
Solution of Balance Equations:

Computing $P_0$:

$$\sum_{n} P_n = 1$$

$$P_0 + P_1 + P_2 + \ldots + P_s + P_{s+1} + \ldots + P_n + P_{n+1} + \ldots = 1$$

$$P_0 + \rho P_0 + (\rho^2/2) P_0 + (\rho^3/3!) P_0 + \ldots + (\rho^s/s!) P_0 + \ldots + (\rho^k/s! s^{k-s}) P_0 = 1$$

$$P_0 [ 1 + \rho + (\rho^2/2) + (\rho^3/3!) + \ldots + (\rho^s/s!) + \ldots + (\rho^k/s! s^{k-s}) ] = 1$$

$$P_0 = [ 1 + \rho + (\rho^2/2) + (\rho^3/3!) + \ldots + (\rho^s/s!) + \ldots + (\rho^k/s! s^{k-s}) ]^{-1}$$
4.4 M/M/s/k Queueing Model

Steady-State Distribution
Solution of Balance Equations:

Computing $P_0$:

$$P_0 = \left[ \sum_{n=0}^{s-1} \frac{\rho^n}{n!} + \sum_{n=s}^{k} \frac{\rho^n}{s! s^{n-s}} \right]^{-1}$$

$P_n$ is a function of $P_0$
if $P_0 = 0 \Rightarrow$ system is infinite
system is finite if an only if $P_0 > 0$

$P_0 > 0$ and finite always for any $\lambda$, $\mu$ and $s$

No Steady-State Condition on $\lambda$, $\mu$, and $s$
4.4 M/M/s/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:

\[ P_n = \frac{\lambda^n}{n! \mu^n} P_0 \quad n < s \]

\[ P_n = \frac{\lambda^n}{s! s^{n-s} \mu^n} P_0 \quad s \leq n \leq k \]

\[ P_0 = \left[ \sum_{n=0}^{s-1} \frac{\rho^n}{n!} + \sum_{n=s}^{k} \frac{\rho^n}{s! s^{n-s}} \right]^{-1} \]

\[ P_n = \frac{\rho^n}{n!} P_0 \quad n < s \]

\[ P_n = \frac{\rho^n}{s! s^{n-s}} P_0 \quad s \leq n \leq k \]
4.4 M/M/s/k Queueing Model

Performance Measures
In steady state

\[ L_B = E[\text{busy servers}] = E[\#\text{Cust. in service}] \]
\[ L_s = L_q + L_B \]
\[ W_s = W_q + \left(\frac{1}{\mu}\right) \]
\[ L_s = \lambda_e W_s \]
\[ L_q = \lambda_e W_q \]
\[ L_B = \lambda_e W_B \]

BP = Blocking Probability

Know 4 measures \Rightarrow all measures are known
4.4 M/M/s/k Queueing Model

Performance Measures

1. Blocking Probability BP:
BP = Pr{system is full} = P_k

2. Effective Arrival Rate $\lambda_e$:
$\lambda_e = \text{arrival rate} \cdot \Pr\{\text{system is not full}\}$

$= \lambda \cdot \Pr\{ n \leq k \}$

$= \lambda \cdot (1 - \text{BP}) = \lambda \cdot (1 - P_k)$
4.4 M/M/s/k Queueing Model

Performance Measures

3. Average Customers in System \( L_s \):

\[
L_s = \sum_{n=0}^{\infty} n \cdot P_n = \sum_{n=0}^{s-1} n \cdot P_n + \sum_{n=s}^{k} n \cdot P_n
\]

\[
= \sum_{n=0}^{s-1} n \frac{\rho^n}{s!} P_0 + \sum_{n=s}^{k} n \frac{\rho^n}{s! s^{n-s}} P_0
\]

\[
= \frac{P_0}{s!} \left( \sum_{n=0}^{s-1} n \cdot \rho^n + s^s \sum_{n=s}^{k} n \frac{\rho^n}{S^n} \right)
\]
4.4 M/M/s/k Queueing Model

Performance Measures

3. Average Busy servers $L_B$:

$$L_B = E[\text{busy servers}] = E[\text{#Cust. in service}]$$

$$L_B = 0.P_0 + 1.P_1 + 2.P_2 + \ldots + s.P_s + s.P_{s+1} + s.P_{s+2} + \ldots$$

$$L_B = 0.P_0 + 1.P_1 + 2.P_2 + \ldots + s \left( \Pr\{ n \geq s \} \right)$$

$$L_B = 0.P_0 + 1.P_1 + 2.P_2 + \ldots + s \left( 1 - \Pr\{ n < s \} \right)$$

$$L_B = \sum_{n=0}^{s-1} n \cdot P_n + s \left( 1 - \sum_{n=s}^{k} P_n \right)$$
4.4 M/M/s/k Queueing Model

Performance Measures

4. Average Busy servers $L_B$:

$$L_B = E[\text{busy servers}] = E[\#\text{Cust. in service}]$$

Using Little’s Formula:  
$$L_B = \lambda_e W_B$$

$W_B = \text{Average time server is busy} = E[S] = 1/\mu$

$$\Rightarrow L_B = \lambda_e W_B = \lambda_e / \mu$$
4.4 M/M/s/k Queueing Model

Performance Measures

5. Average Customers in Queue $L_q$:

$$L_q = L_s - L_B$$

or

Number of customer in queue = 0 with $Pr\{n \leq s\} = P_0 + P_1 + P_2 + \ldots + P_s$

Number of customer in queue = 1 with $Pr\{n = s+1\} = P_{s+1}$

Number of customer in queue = 2 with $Pr\{n = s+1\} = P_{s+2}$

....

$$L_q = 0.(P_0 + P_1 + P_2 + \ldots + P_s) + 1.P_{s+1} + 2.P_{s+2} + 2.P_{s+3} \ldots$$

$$= \sum_{n=s}^{\infty} (n-s) \cdot P_n = L_s - \frac{\lambda e}{\mu}$$
Performance Measures

6. Utilization of the System $U$:

$$U = Pr\{ n > 0 \} = P_1 + P_2 + P_3 + \ldots + P_k = 1 - P_0$$

7. Utilization of the Service $SU$:

$$SU = Pr\{ \text{all servers busy} \} = Pr\{ n \geq s \}$$

$$SU = 1 - (P_0 + P_1 + P_2 + \ldots + P_{s-1})$$
4.4 M/M/s/k Queueing Model

Performance Measures

8. Average Time Spent in System $W_s$:

$$L_s = \lambda_e \cdot W_s \quad \Leftrightarrow \quad W_s = \frac{L_s}{\lambda_e}$$

9. Average Waiting time in Queue $W_q$:

$$L_q = \lambda_e \cdot W_q \quad \Leftrightarrow \quad W_q = \frac{L_q}{\lambda_e}$$
4.4 M/M/s/k Queueing Model

Example:

A barber shop has 3 barbers and a total of 7 waiting seats. Interarrival times are exponentially distributed, and an average of 20 prospective customers arrive each hour at the shop. Those customers who find the shop full do not enter. The barber takes an average of 12 minutes to cut each customer's hair. Haircut times are exponentially distributed.

1. If you arrive to the shop, what is the probability that you get served?
2. On average how many hair cuts is performed per hour?
3. If you enter the shop at 9:00 am when do you expect to leave?
4. What is the probability that a customer waits if he enters?
5. What is the average empty seats in the shop?
6. The owner of the shop decided to accept any arrival to enter the shop, what is the minimum he must hire?
4.4 M/M/s/k Queueing Model

Example:

Arrivals: $\lambda = 20 \text{ cust/hr}$ Poisson
Service: $E[S] = 12 \text{ min.}$ Exponential

$\Rightarrow \mu = 1/E[S] = 1/12 \text{ cust/min}$

$= 60/12 = 5 \text{ cust/hr}$

Number of Servers: $s = 3$
System Size = waiting seats + service seats = $7 + 3 = 10$

$\Rightarrow M/M/3/10$

$\rho/s = 20/(3(5)) = 1.33$
4.4 M/M/s/k Queueing Model

**Example:**
1. If you arrive to the shop, what is the probability that you get served?

\[
\Pr\{\text{get served}\} = \Pr\{\text{enter}\} = \Pr\{n < 10\} = 1 - P_{10}
\]

\[
P_0 = \left[\frac{\sum_{n=0}^{s-1} \rho^n}{n!} + \sum_{n=s}^{k} \frac{\rho^n}{s! s^{n-s}}\right]^{-1}
\]

\[
P_n = \frac{\rho^n}{n!} P_0 \quad \text{for } n < s
\]

\[
P_n = \frac{\rho^n}{s! s^{n-s}} P_0 \quad \text{for } s \leq n \leq k
\]

<table>
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<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
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<td>0.0033</td>
<td>0.0133</td>
<td>0.0266</td>
<td>0.0355</td>
<td>0.0473</td>
<td>0.0631</td>
<td>0.0841</td>
<td>0.1121</td>
<td>0.1495</td>
<td>0.1993</td>
<td>0.2658</td>
</tr>
</tbody>
</table>

\[
\Pr\{\text{get served}\} = 1 - 0.2685 = 0.7315
\]
4.4 M/M/s/k Queueing Model

Example:

2. On average how many haircuts is performed per hour?
   Average number of hair cuts is performed per hour = $\lambda_e$
   
   $$\lambda_e = \lambda (1 - P_{10}) = 20 \times (0.7315) = 14.684 \text{ haircut/hr.}$$

3. If you enter the shop at 9:00 am when do you expect to leave?
   
   E[Departure time] = 9:00 + $W_s$
   
   $$W_s = \frac{L_s}{\lambda_e} = \frac{(\sum nP_n)}{\lambda_e} = \frac{7.62}{14.684} = 0.5186 \text{ hr} = 0:32$$
   
   E[Departure time] = 9:00 + $W_s = 9:00 + 00:32 = 9:32$$
4.4 M/M/s/k Queueing Model

Example:

4. What is the probability that a customer waits if he enters?

\[ \Pr\{\text{waiting}\} = \Pr\{3 \leq n < 10\} = P_3 + P_4 + \ldots + P_9 = 0.691 \]

5. What is the average empty seats in the shop?
\[
E[\# \text{ empty seats}] = \text{waiting seats} - E[\# \text{ waiting customers}]
\]
\[
L_q = L_s - \frac{\lambda_e}{\mu} = 7.6154 - \frac{(14.684)}{5} = 4.6786
\]
\[
E[\# \text{ empty seats}] = 7 - 4.686 = 2.2314 \text{ seats}
\]
4.4 M/M/s/k Queueing Model

Example:
6. The owner of the shop decided to accept any arrival to enter the shop, what is the minimum barber he must hire?

Accept any arrival ⇒ system size = k → ∞
The new system is M / M /s
⇒ ρ/s must be less than 1

\[ \rho = 4 \Rightarrow \frac{4}{s} < 1 \Rightarrow s > 4 \]

s = 5, 6, 7, ... ⇒ minimum number = 5 barbers