ON RECOVERING OF A BIVARIATE POLYNOMIAL FROM ITS RADON PROJECTIONS

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Abstract

For a given set of angles $\theta_0 < \cdots < \theta_n$ in $[0, \pi)$, we give explicitly a set of $k+1$ parallel X-ray beams in each direction θ_k, $k = 0, \ldots, n$, which can be used to reconstruct a bivariate polynomial in Π_n^2 uniquely from a set of $\binom{n+2}{2}$ Radon projections in the above directions. Also we give an outline of an algorithm which can be used to reconstruct the polynomial. Some numerical results are given too.

1 Introduction

In many practical problems the information of a bivariate function comes from the values of its integrals along a finite set of line segments, for example in tomography and electronic microscopy. This motivates many authors to study the problem of approximating bivariate functions using this type of information. See, for example, [4], [5], [7], [9], [10], [12], and the bibliography therein.

In [10], Marr proved that a bivariate polynomial of total degree at most n can be reconstructed by using its integrals alon a finite number of chords joining equidistant points on the unite circumference.

A particular case of a general multivariate result was given by Hakopian in [6] (see also [4]), shows that a bivariate polynomial of total degree $\leq n$ can be recovered uniquely from its integrals over a set of $\binom{n+2}{2}$ chords joining any given $n+2$ distinct points on the boundary of the unit disk. Another result, based

the solution of the interpolation problem considered in [1]. First we need some preliminaries.

We shall denote by Π_n^2 the set of all real algebraic polynomials in two variables of total degree $\leq n$. If $P \in \Pi_n^2$, then

$$P(x,y) = \sum_{i+j\leq n} \alpha_{ij} x^i y^j \ , \ \alpha_{ij} \in \mathbb{R}.$$

It is known that the dimension of Π_n^2 is $\binom{n+2}{2}$. Let $\mathbf{B} := \{\mathbf{x} := (x,y) \ : \ \|\mathbf{x}\| \leq 1\}$ be the unit disk on the plane $\mathbb{R}^2$, where $\|\mathbf{x}\|^2 = (x^2+y^2)$. For a given real valued function f defined in $\mathbb{R}^2$ we shall assume that the integrals of f are known along all line segments on the unit disk $\mathbf{B}$. Given $t \in [-1,1]$ and an angle $\theta \in [0,\pi)$, measured counterclockwise from the positive x axis, the equation

$$\ell(x,y) := x\cos\theta + y\sin\theta - t = 0$$

defines the line ℓ which passes through the point $(t\cos\theta, t\sin\theta)$ and is perpendicular to the vector $\langle \cos\theta, \sin\theta\rangle$. Let $I(\theta,t)$ be the set of points of intersection of the line ℓ and the unit disk $\mathbf{B}$. i.e.,

$$I(\theta,t) := \ell \cap \mathbf{B}, \ \ \theta \in [0,\pi), \ \ t \in [-1,1].$$

It is easy to see that the points (x,y) on $I(\theta,t)$ are given by

$$x = t\cos\theta - s\sin\theta, \quad y = t\sin\theta + s\cos\theta$$

where $|s| \leq \sqrt{1-t^2}$.

For a given $t \in [-1,1]$ and $0 \leq \theta < \pi$, the *Radon projection* of a function f along the line segment $I(\theta,t)$ is given by

$$\begin{aligned} \mathcal{R}_\theta(f;t) \ &:= \ \int_{I(\theta,t)} f(\mathbf{x})\, d\mathbf{x} \\ &= \ \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds. \end{aligned}$$

It is also called an X-ray. Since $I(\theta,t) \equiv I(\theta+\pi,-t)$ then

Since the function $f \equiv 0$ has all its projections equal to zero, it follows that the only function which has the zero Radon transform is the zero function.

In fact,(see [2]), if $P \in \Pi_n^2$ then for each fixed θ there exists a univariate polynomial p of degree at most n such that

$$\mathcal{R}_\theta(P;t) = \sqrt{1-t^2}\; p(t), \quad -1 \leq t \leq 1.$$

Moreover every algebraic polynomial $P \in \Pi_n^2$ is uniquely determined by only a finite number of projections [11]:

Theorem A. *Let $\theta_0 < \cdots < \theta_n$ be any given angles in $[0,\pi)$. Then the projections*

$$\mathcal{R}_{\theta_k}(P;t), \quad -1 \leq t \leq 1, \quad k = 0,\ldots,n,$$

determine P uniquely.

The problem of recovering a bivariate polynomial $P \in \Pi_n^2$ by using a finite number of X-rays, matching the dimension of Π_n^2 was considered in [1]. The X-rays were taken along a finite set of chords $J = \{I(\theta,t)\}$ where the cardinality of J is equal to the dimension of Π_n^2. The set J was partitioned into $n+1$ supgroups, such that the kth supgroup consists of $k+1$ parallel line segments in the unit ball $\mathbf{B}$.

The problem was described by a set of given angles $\theta_0 < \cdots < \theta_n$ in $[0,\pi)$ and a triangular matrix $T = \{t_{ki}\}$ of points

$$t_{kk} < \cdots < t_{kn}, \quad k = 0,\ldots,n,$$

associated with these angles. Here we consider the problem of characterizing the locations of a set of nodes $\{t_{kj}\}$ for which the interpolation of the data $\{\mathcal{R}_{\theta_k}(\cdot;t_{kj})\}$ by polynomials of degree at most n is poised.

Let

$$U_m(t) := \frac{\sin(m+1)\theta}{\sin\theta}, \quad t = \cos\theta \tag{1}$$

be the Chebyshev polynomial of the second kind of degree m.

Consider the matrices

Theorem B. *For given angles* $0 \le \theta_0 < \cdots < \theta_n < \pi$ *and associated points* $T = \{t_{ki}\}_{k=0,i=k}^{n\ \ n}$*, the interpolation problem*

$$\int_{I(\theta_k,t_{ki})} P(\mathbf{x})\,d\mathbf{x} = \gamma_{ki}, \quad k = 0,\ldots,n,\ i = k,\ldots,n, \quad P \in \Pi_n^2, \tag{2}$$

is poised if and only if

$$\det \mathbf{U}_k \neq 0 \quad \text{for} \quad k = 1,\ldots,n.$$

The martices $\mathbf{U}_k$ are not invertible for all choices of $T = \{t_{ki}\}_{k=0,i=k}^{n\ \ n}$. Indeed, for $k = n$ we have $\det \mathbf{U}_k = U_n(t_{nn})$ which is zero if t_{nn} is a zero of U_n. However for each fixed k the determinant of $\mathbf{U}_k$ can be considered as a polynomial in the $span\{U_k,\ \ldots,\ U_n\}$, thus it is different than zero for almost all choices of $\{t_{ki}\} \in \mathbb{R}$. But given a set of points $\{t_{ki}\}$ it is often difficult to determine if it gives a unique solution of the problem (2).

2 Main Result

For a given positive integer n let $\alpha = \frac{\pi}{n+2}$, and let $\eta_i := \cos\left(\frac{i\pi}{n+2}\right) = \cos(i\alpha),\ i = 1,\ldots,n+1$, be the zeros of the polynomial $U_{n+1}(x)$. We have:

Theorem 1. *Let* $t_{ki} = \eta_{i+1}$ *for* $i = k,\ldots,n,\ \ k = 0,\ldots,n$. *Then* $\det \mathbf{U}_k \neq 0$ *for* $k = 1,\ldots,n$ *and consequently the interpolation problem (2) is poised.*

Proof. In view of (1), by direct evaluation, it follows that:

$$U_{n-j}(t_{k,n-l}) = (-1)^{n-l-j}\frac{\sin\{(j+1)(l+1)\}}{\sin\{(n-l+1)\alpha\}},$$

for $l = 0,\ldots,n-k, j = 0,\ldots,n$.
From the rows of $\det \mathbf{U_k}$ we factor out $\frac{1}{\sin\{(n-l+1)\alpha\}}$ for each $l = 0,\ldots,n-k$, then what is left is a symmetric determinant. If n is even we factor out (-1) from the even rows of the resulting determinant to get that all elements in each column have the same sign, thus we factor out (-1) from its even columns. This will result in a factor of $\pm 1 := \varepsilon$. (If n is odd we factor out (-1) from the odd rows

then we have

$$\det \mathbf{U}_k = \frac{\varepsilon}{\prod_{j=1}^{m} \sin\{(n-j+2)\alpha\}} \det \mathbf{U}_k^*$$

We shall show that

$$\det \mathbf{U}_k^* = 2^{m(m-1)/2} \prod_{j=1}^{m} \sin(j\alpha) \det \mathbf{V}_k,$$

where

$$\det \mathbf{V}_k := \begin{pmatrix} 1 & \cos\alpha & \cdots & (\cos\alpha)^{m-1} \\ 1 & \cos(2\alpha) & \cdots & (\cos(2\alpha))^{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cos(m\alpha) & \cdots & (\cos(m\alpha))^{m-1} \end{pmatrix},$$

Indeed, by the formula: $e^{i\phi} = \cos\phi + i\sin\phi$, it follows that

$$\cos(n\phi) + i\sin(n\phi) = (\cos\phi + i\sin\phi)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} (i\sin\phi)^{\mu} (\cos\phi)^{n-\mu} \tag{3}$$

Comparing the imaginary parts in both sides of (3) we get:

$$\sin((2m)\phi) = \sin\phi \sum_{j=1}^{m} \beta_j \cos^{2j-1}\phi,$$

$$\sin((2m+1)\phi) = \sin\phi \sum_{j=0}^{m} \alpha_j \cos^{2j}\phi,$$

where α_i, β_i are constants. Hence for the general element in $\mathbf{U}_k^*$ we have

$$u_{l,2j}^* = \sin(l\alpha) \sum_{\nu=1}^{j} a_\nu \cos^{2\nu-1}(l\alpha), \tag{4}$$

$$u_{l,2j-1}^* = \sin(l\alpha) \sum_{\nu=0}^{j} b_\nu \cos^{2\nu}(l\alpha), \tag{5}$$

for some constants a_ν, b_ν. Let us mention that the leading coefficients in (4),

where $\mathbf{U}'_k$ is the matrix with general element $u'_{i,j}$ given by

$$u'_{l,2j} = \sum_{\nu=1}^{j} a_\nu \cos^{2\nu-1}(l\alpha), \tag{6}$$

$$u'_{l,2j-1} = \sum_{\nu=0}^{j} b_\nu \cos^{2\nu}(l\alpha). \tag{7}$$

Thus, for each k, $u'_{l,k}$ is a polynomial of degree $k-1$ in $\cos(l\alpha)$, which is even or odd according to k.

For a fixed j we introduce the notation $C_{l,j} := \left[u'_{l,j}\right], l = 1, ..., m$, to denote the jth column of $\mathbf{U}'_k$. In view of (6), (7) we have $C'_{l,1} := C_{l,1} = [1]$, and $C_{l,2} = [a_1 \cos(l\alpha)]$. Factoring out a_1 from $C_{l,2}$, the second column of $\mathbf{U}'_k$ becomes: $C'_{l,2} = [\cos(l\alpha)]$. Since $C_{l,3} = [b_2 \cos^2(l\alpha) + b_1]$, we subtract $b_1 \times C'_{l,1}$ from $C_{l,3}$, then we factor out the coefficient b_2, we find that the 3rd column of $\mathbf{U}'_k$ becomes $C'_{l,3} = [\cos^2(l\alpha)]$.

Continuing in this way, by subtracting from each odd column the multiples of the preceding odd columns and subtracting from each even column the multiples of the preceding even columns then factoring out the leading coefficients a_j, b_j we reduce all the columns of $\mathbf{U}'_k$ to the form $C'_{l,\mu} = [\cos^{\mu-1}(l\alpha)], \mu = 1, \ldots, m; l = 1, \ldots, m$.

Since adding multiples of columns of a determinant to another column of the same determinant will not affects its value, therefore

$$\begin{aligned}
\det \mathbf{U}'_k &= \prod_{j=1}^{[m/2]} a_j \prod_{j=0}^{[(m-1)/2]} b_j \det \mathbf{V}_k \\
&= 2^{1+2+\cdots+(m-1)} \det \mathbf{V}_k = 2^{m(m-1)/2} \det \mathbf{V}_k.
\end{aligned}$$

Thus, from the above steps we finally get that

$$\det \mathbf{U}_k = \varepsilon \, 2^{m(m-1)/2} \prod_{j=1}^{m} \frac{\sin(j\alpha)}{\sin\{(n-j+2)\alpha\}} \det \mathbf{V}_k.$$

It is evident that $\det \mathbf{V}_k$ is a Vandermonde determinant, thus we have

Now we will give the steps that can be used to reconstruct the interpolating polynomial.

Every polynomial $P \in \Pi_n^2$ can be represented as

$$P(x,y) = \sum_{p+q \le n} a_{pq} x^p y^q. \tag{8}$$

Step 1. Compute the Radon projections of the monomials $x^p y^q$:

$$\int_{I(\theta_k,t_{ki})} x^p y^q \, dx\, dy = \int_{-\sqrt{1-\eta_{i+1}}}^{\sqrt{1-\eta_{i+1}}} (\eta_{i+1}\cos\theta_k - s\sin\theta_k)^p (\eta_{i+1}\sin\theta_k + s\cos\theta_k)^q \, ds.$$

Step 2. Solve the following linear system for the coefficients $\{a_{pq}\}$:

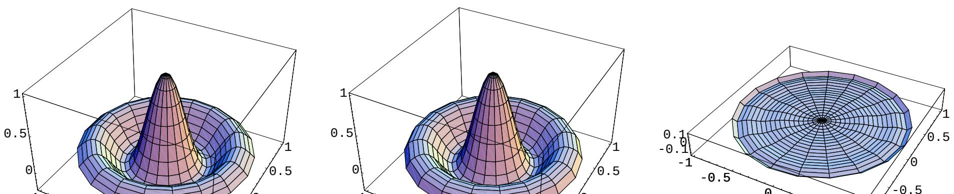
$$\sum_{p+q\le n} a_{pq} \int_{I(\theta_k,t_{ki})} x^p y^q \, dx\, dy = \gamma_{ki}, \text{ for } i = k,\dots,n, \; k = 0,\dots,n.$$

Step 3. Substitute the values of the coefficients $\{a_{pq}\}$ in (8).

We apply the above algorithm for two functions that are usually used for testing new methods for recovering bivariate functions: namely, to the so called "Mexican hat":

$$\frac{\sin\left(3\pi\sqrt{x^2+y^2+10^{-18}}\right.}{3\pi\sqrt{x^2+y^2+10^{-18}}}, \; x^2+y^2 \le 1$$

and to the cone: $\sqrt{x^2+y^2}$, $x^2+y^2 \le 1$. We get the following numerical results.



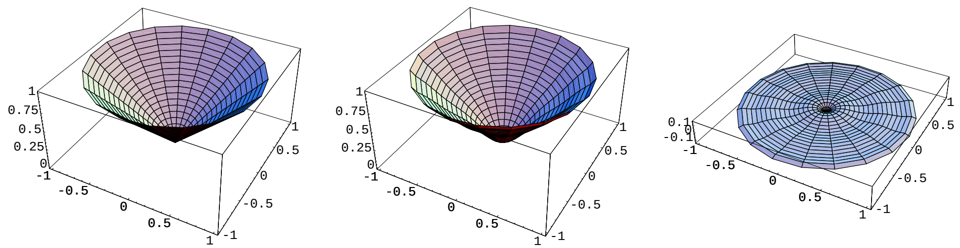


Figure 2: The graphs of the cone, its interpolating polynomial of degree 10 and the error.

The uniform norms of the error are given in the following table.

n	Mexican hat	the cone
8	0.136998	0.0702261
10	0.0122434	0.0582879

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