KING SAUD UNIVERSITY
College of Sciences
Department of Mathematics

Final Examination / Semester II/ 1427/1428

Math 580

Question 1.

1) State the dominated convergence theorem. Prove that if f is integrable on $[0,1]$, then $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$

2) We define the sequence $(I_n)_n$ by $I_n = \int_0^{\frac{\pi}{4}} tg^n(x)\ dx$.

a) Give the limit of the sequence $(I_n)_n$. Find I_0 and I_1. Prove that for $n \geq 1$, $I_n + I_{n+2} = \dfrac{1}{n+1}$.

b) Deduce the values of $\sum_{k=0}^{+\infty} \dfrac{(-1)^k}{2k+1}$ and $\sum_{k=1}^{+\infty} \dfrac{(-1)^k}{k}$.

Question 2. Let $(X, \mathcal{T})$ be a measurable space, $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X twise differents and let $(m_n)_{n\in\mathbb{N}}$ be a sequence of real strictly positive numbers. For $A \in \mathcal{T}$, we define $\mu(A) = \sum_{x_i \in A} m_i$.

1) Show that μ is a measure on X.
2) Which are the null subsets for the measure μ.
3) Show that if for any n; $\{x_n\} \in \mathcal{T}$, then the completed σ-algebra of X for the measure μ is $\mathcal{P}(X)$.
4) Let $A = \{1/n; n \geq 1\}$.

a) Find $\mathcal{T}$ the σ-algebra on $\mathbb{R}$, generated by $\mathcal{P}(A)$.

b) For $E \in \mathcal{T}$, we define $\mu(E)$ as the number of elements of $E \cap A$.

c) Show that μ is a measure on $\mathcal{T}$.

d) Find the completion of $(\mathbb{R}, \mathcal{T}, \mu)$.

Question 3.

1) Let $f \in L^1(\mathbb{R})$ and let $\alpha > 0$. Show that $\lim_{n\to+\infty} \dfrac{f(nx)}{n^\alpha} = 0$, for almost $x \in \mathbb{R}$ (Hint : we can integrate the series $\sum_{n=1}^{+\infty} \dfrac{f(nx)}{n^\alpha}$ on $\mathbb{R}$).

2) Let $(X, \mathcal{B}, \mu)$ be a measure space and f be a nonnegative measurable function on X with values in $\mathbb{R}$. Let $\delta > 1$, for all $n \in \mathbb{Z}$ we define the set

$$E_n = \{x \in X : \delta^n < f(x) \leq \delta^{n+1}\}.$$

Show that f is integrable if and only if

$$\sum_{n=-\infty}^{+\infty} \delta^n \mu(E_n) < +\infty.$$

Question 4. 1) State the Radon-Nikodym theorem.

2) Let $(X, \mathcal{B}, \mu)$ be a measure space. Show that if $\psi(E) = \int_E f d\mu$ where $\int f d\mu$ is defined , then ψ is a signed measure.

3) For $E \subset \mathbb{R}$, we define $\nu(E) = \int_E 2xe^{-x^2}$. What are the positive, negative and null sets with respect to ν? Given a Hahn decomposition of $\mathbb{R}$ with repect to ν.

Question 5. 1) We consider the function F defined by :

$$F(x) = \int_0^{+\infty} \frac{e^{-xt}}{1+t^2} dt.$$

a) Find $\lim_{x \to +\infty} F(x)$ and $\lim_{x \to 0} F(x)$.

b) Prove that F is of class C^2 for $x > 0$ and find $F''(x)$.

2) Prove that $\int_0^{+\infty} \frac{\cos x}{e^x + 1} dx = \sum_{n=1}^{+\infty} \frac{n(-1)^n}{n^2+1}$.