KING SAUD UNIVERSITY
College of Sciences
Department of Mathematics

Mid-Term examination/ Semester II/ 1427/1428

Math 580

Question 1.

1)State and prove the Fatou lemma.

2) Let f be an integrable function on a measure space $(X, \mathcal{B}, \mu)$. Prove that $\{x \in X;\ f(x) = \pm\infty\}$ is a null set.

3) Let $(X, \mathcal{B}, \mu)$ be a measure space and f be an integrable function on X. We assume that $\int_E f(x)d\mu(x) = 0$ for all measurable set E. Show that $f = 0$ a.e.

Question 2. Let λ be the Lebegue measure on $\mathbb{R}$. Compute the following integrals:

1) $\int_{[0,+\infty[} e^{-[x]} d\lambda(x)$, where $[x]$ is the integer part of x.

2) $\int_{[0,\pi]} f(x)d\lambda(x)$, where $f(x) = Cosx$ if $x \in \mathbb{Q} \cap [0, \pi]$ and $f(x) = Sinx$ otherwise.

3) $\int_{[0,1]} \chi_{\mathbb{R}\setminus\mathbb{Q}}(x)d\lambda(x)$.

Question 3.

a) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Prove that if $E \in \mathcal{M} \bigotimes \mathcal{N}$, then the functions $x \in X \to \nu(E_x)$ and $y \in Y \to \mu(E^y)$ are measurable on X and Y respectively, and

$$\mu \times \nu(E) = \int_X \nu(E_x)d\mu(x) = \int_Y \mu(E^y)d\nu(y).$$

(**Recall that* $E_x = \{y \in Y : (x, y) \in E\}$ *and* $E^y = \{x \in X : (x, y) \in E\}$.)

b) Let $X = [0, 1]$, $\mathcal{B}$ the Borel σ-algebra of $[0, 1]$, μ is the Lebesgue measure and ν the counting measure on $\mathcal{B}$ (if $B \in \mathcal{B}$, $\nu(B)$ is the number of elements of B) . Let $D = \{(x, y) \in X \times X : x = y\}$.

i) Show that D is measurable with respect to the σ-algebra $\mathcal{B} \bigotimes \mathcal{B}$.

ii) Show that $\int_0^1 \int_0^1 \mathcal{X}_D(x, y)d\mu(x)d\nu(y) \neq \int_0^1 \int_0^1 \mathcal{X}_D(x, y)d\nu(y)d\mu(x)$.
Explain why these integrals are not equal?