PH.D. COMPREHENSIVE EXAMINATION

College of Science $\qquad$ 2^{end} semester 1426-1427
Department of Mathematics $\qquad$ Time : 3 hours

Answer 5 questions

Section A

Question 1.

a) i) Let f be an analytic function on $\mathbb{C}$. Prove that for any $a, b \in \mathbb{C},\ a \neq b$ we have for $R > \sup(|a|, |b|)$:

$$\frac{1}{2i\pi}\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}dz = \frac{f(a)-f(b)}{a-b}.$$

ii) Prove that if in addition, f is bounded on $\mathbb{C}$, then

$$\frac{1}{2i\pi}\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}dz \to 0,\ \text{when } R \to +\infty.$$

Deduce that any *bounded analytic* function on $\mathbb{C}$ is constant.

b) Prove that the function $f(z) = \dfrac{z-1}{z+1}$ is a conformal mapping from the half-plane $Re(z) > 0$ into the unit disc $|z| < 1$.

Question 2.

For $R > 1$, let γ_R be the half-circle defined by $\gamma_R(t) = Re^{it},\ t \in [0, \pi]$. We consider the function $f(z) = \dfrac{ze^{\mathrm{i}3z}}{(z^2+1)^2}$.

a) Prove that the integral $\displaystyle\int_{\gamma_R} f(z)dz \to 0$ when $R \to +\infty$.

b) Use the residue theorem to find the value of the integral $\displaystyle\int_0^{+\infty} \frac{x\sin 3x}{(x^2+1)^2}dx$

Section B

Question 3.

a) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F,\ E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$.

i) Show that $\overline{\mathcal{M}}$ is a σ-algebra.

ii) Verify that the extension $\overline{\mu}$ of μ on $\overline{\mathcal{M}}$ is a complete measure.

b) i) State the definition of an outer measure.

ii) Let X be a space. We consider $\mathcal{M} \subset \mathcal{P}(X)$ an algebra of sets and f a non-negative function defined on $\mathcal{M}$, such that $f(\emptyset) = 0$. For any $A \subset X$, define

$$\mu(A) = \inf\{ \sum_{i=1}^{\infty} f(E_i) : E_i \in \mathcal{M} \text{ and } A \subset \bigcup_{1 \leq i \leq \infty} E_i \}.$$

Show that μ is an outer measure.

c) If μ_1,...,μ_n are measures on $(X, \mathcal{M})$ and $a_1, ..., a_n$ positive numbers. Prove that $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Question 4.

a) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Prove that if $E \in \mathcal{M} \bigotimes \mathcal{N}$, then the functions $x \in X \rightarrow \nu(E_x)$ and $y \in Y \rightarrow \mu(E^y)$ are measurable on X and Y respectively, and

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

(**Recall that* $E_x = \{y \in Y : (x, y) \in E\}$ *and* $E^y = \{x \in X : (x, y) \in E\}$.)

b) Let $X = [0, 1]$, $\mathcal{B}$ the Borel σ-algebra of $[0, 1]$, μ is the Lebesgue measure and ν the counting measure on $\mathcal{B}$ (if $B \in \mathcal{B}$, $\nu(B)$ is the number of elements of B) . Let $D = \{(x, y) \in X \times X : x = y\}$.

i) Show that D is measurable with respect to the σ-algebra $\mathcal{B} \bigotimes \mathcal{B}$.

ii) Show that $\displaystyle\int_0^1 \int_0^1 \mathcal{X}_D(x, y) d\mu(x) d\nu(y) \neq \int_0^1 \int_0^1 \mathcal{X}_D(x, y) d\nu(y) d\mu(x)$. Explain why these integrals are not equal?

Section C

Question 5.

a) i) State the definition of a Banach space.

ii) Let $\mathcal{C}([0, 1], \mathbb{R})$ be the real vector space of continuous functions defined from $[0, 1]$ into $\mathbb{R}$. For $g \in \mathcal{C}([0, 1], \mathbb{R})$, we consider the norm $||g||_\infty = \sup_{x \in [0,1]} |g(x)|$. Prove that $(\mathcal{C}([0, 1], \mathbb{R}), ||.||_\infty)$ is a Banach space.

b) Let $\mathcal{C}^1([0, 1], \mathbb{R})$ be the real vector space of functions of class C^1 defined from $[0, 1]$ into $\mathbb{R}$.

i) Prove that the function $N : f \rightarrow ||f'||_\infty + |f(0)|$ is a norm on $C^1([0, 1], \mathbb{R})$.

ii) Show that $(\mathcal{C}^1([0, 1], \mathbb{R}), N)$ is a Banach space (we can use the question a)-ii).

Question 6.

If A, B, C and D are pairwise commutative operators on a Hilbert space H. Show that the necessary and sufficient condition for the operator matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to be invertible is that the formal determinant $AD - BC$ be invertible.