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MID-TERM EXAMINATION
Math-685

**Exercise 1.** Let $f : D \to D'$ be a proper holomorphic map between domains in $\mathbb{C}^n$. Prove that if $u \in PSH(D)$, then the function

$$v(z) = max\{u(w) : w \in f^{-1}(z)\}$$

defines a plurisubharmonic function on $D'$.

**Exercise 2.**

**I**

1- State the definition of the Caratheodory metric and the Kobayashi metric.

2- Prove that if $D$ is a bounded domain in $\mathbb{C}$ and $z_0 \in D$ such that $F_c^D(z_o, 1) = F_k^D(z_o, 1) \neq 0$, then $D$ is biholomorphic to the unit disc.

3- Let $D$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with boundary of class $C^2$.

a) Prove that there exists $C_1 > 0$ such that for all $z \in D$,

$$F_k^D(z, v) \leq \frac{C_1|v|}{dist(z, \partial D)}.$$

*Let $r$ be a strongly plurisubharmonic function, defining the domain $D$, $p \in D$, and $f : \Delta(0, 1) \to D$ be a holomorphic function such that $f(0) = p$ and $f'(0) = cv$ ($c \in \mathbb{C}$ and $v \in \mathbb{C}^n$).* Let $q \in \partial D$ be the nearest point to $p$.

b) Justify the existence of a constant $a > 0$ such that the function $\rho(z) = r(z) - a|z - q|^2$ is strongly plurisubharmonic.

c) Justify that we may assume without loss of generality that $f$ is continuous on $\overline{\Delta(0, 1)}$.

d) Prove that $\frac{a}{2\pi}\int_0^{2\pi} |f(e^{i\theta}) - q|^2 d\theta \leq a[dist(p, \partial D)]^2 - r(p)$ (Hint: consider the function $u = \rho \circ f$)

e) Deduce that there exists a constant $C_2^1 > 0$ such that

$$\int_0^{2\pi} |f(e^{i\theta}) - q|^2 d\theta \leq C_2^1 dist(p, \partial D).$$

f) Use Cauchy's formula to prove that $|f'(0)| \leq C_2^2[dis(p, \partial D)]^{\frac{1}{2}}$ for some positive constant $C_2^2$.

g) Deduce the existence of a constant $C_2 > 0$ such $F_k^D(p, v) \geq \dfrac{C_2|v|}{[dis(p, \partial D)]^{\frac{1}{2}}}$

## II

1- State the definition of Hopf's lemma.

2- Prove that if $D$ is a bounded domain with boundary of class $C^2$ and $r$ a defining function of $D$ , then there exists a constant $A > 0$ such that for all $\xi \in \partial D$, $|\dfrac{\partial r}{\partial n_\xi}(\xi)| > A$, where $n_\xi$ denotes the normal vector to the boundary of $D$ at $\xi$ and $\dfrac{\partial r}{\partial n_\xi}$ is the partial derivative in the direction of $n_\xi$ . Prove that this statement is equivalent to the Hopf lemma.

*Let $g : D \to D'$ be a proper holomorphic map between bounded strongly pseudoconvex domains in $\mathbb{C}^n$.*

3- Prove that there exists a constant $C_3 > 0$ such for all $z \in \partial D$,

$$\frac{1}{C_3} dist(z, \partial D) \leq dist(g(z), \partial D') \leq C_3 dist(z, \partial D).$$

4- Prove there exists a constant $C > 0$ that for all $z \in D$ and $v \in \mathbb{C}^n$,

$$|g'(z)v| \leq \frac{C|v|}{[dist(z, D)]^{\frac{1}{2}}}.$$

5- Deduce that $g$ is continuous on $\overline{D}$. □