King Saud University
College of Sciences
Department of Mathematics

I-Semester 1427-1428
Time: 3 hours

Final Examination
Math 683

Exercise 1.(35 mn)

1) State the maximum principle.
Prove that if Ω is connected and f takes real values in the circle $\{z; |z-z_o| = R\} \subset \Omega$, then f is constant.

2) Let U be a harmonic function of class C^2 on a simply connected domain D in $\mathbb{C}$. Consider $f(z) = \frac{\partial U}{\partial x}(z) - i\frac{\partial U}{\partial y}(z)$.

a) Prove that f is holomorphic on D.

b) Justify that f admits a primitive on D. Deduce the existence of a holomorphic function g on D such that $U = \mathcal{R}e(g)$ on D.

3) Precise the image of the half-plane $\{z \in \mathbb{C} : Rez > 0\}$ by the mapping $f(z) = z^2$ and give a conformal transformation from $\mathbb{C}\backslash\mathbb{R}$ into the unit disc.

4) If Ω is a simply connected domain in $\mathbb{C}$, justify the non-existence of a conformal transformation from $\mathbb{C}$ into Ω.

Exercise 2.(25 mn)

1) Let $a, b, c \in D(0,1)$ Prove that the function

$$f(z) = z(\frac{z-a}{1-\bar{a}z})^n(\frac{z-b}{1-\bar{b}z})^m + c$$

has exactly $n+m+1$ roots in $D(0,1)$ (use Rouché theorem).

2) State the Cauchy inequality and find all holomorphic mappings $f : \mathbb{C} \to \mathbb{C}$ that satisfy $|f(z)| \leq |z|$ (Hint: note that $f(0) = 0$).

3) let $f(z) = \prod_{n=1}^{\infty} cosh(\frac{z}{n})$. Prove that f is holomorphic on $\mathbb{C}$.

Exercise 3.(1 hour)

We denote by D the unit disc. For $a, b \in D$, we define $\delta(a,b) = |\frac{b-a}{1-\bar{a}b}|$.

1) Prove that $\delta(a,b) < 1$ for all $a, b \in D$ (Hint: apply the maximum principle to the function $b \to \frac{b-a}{1-\bar{a}b}$.

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2) Let $f : D \to D$ be a holomorphic function. We denote by $h(z) = \dfrac{z-a}{1-\bar{a}z}$, $k(z) = \dfrac{z-f(a)}{1-\overline{f(a)}\ z}$ and $g = k \circ f \circ h^{-1}$.

a) Verify that g satisfies the hypothesis of Schwarz lemma.

b) Apply the Schwarz lemma to the function g to deduce that $\delta(f(a), f(b)) \leq \delta(a, b)$ for all $a, b \in D$.

c) Deduce from b) that if moreover f is bijective then $\delta(f(a), f(b)) = \delta(a, b)$ for all $a, b \in D$.

3) Conversely: assume that there exist $a \neq b$ in D such that $\delta(f(a), f(b)) = \delta(a, b)$. Prove that f is bijective from D to D (Hint: use the Schwarz lemma).

<u>Exercise 4.</u>(1 hour)

Let $f(z) = cotan(\pi z)$.

1) Find the poles of the function f and their correspondent residues .

2) a) Let C_n the square contour with corners at $\pm(n+\frac{1}{2}) \pm i(n+\frac{1}{2})$ $n \in \mathbb{N}$. Prove that the function f is bounded on ∂C_n (the boundary of C_n) by a constant independent of n.

b) Use the Residues theorem to prove that for all $z \in \mathbb{C}\backslash\mathbb{Z}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{1}{2\pi i}\int_{\partial C_n} \frac{cotan\pi w}{w-z} dw = cotan\pi z - \frac{1}{\pi}\sum_{p=-n}^{p=n} \frac{1}{z-p}.$$

3) a) Verify that $\displaystyle\int_{\partial C_n^+} \frac{cotan\pi w}{w} dw = -\int_{\partial C_n^-} \frac{cotan\pi w}{w} dw$, where ∂C_n^+denotes the vertical lines of the square and ∂C_n^- denotes the horizontal lines. Deduce that

$$\int_{\partial C_n} \frac{cotan\pi w}{w} dw = 0.$$

b) Deduce from 3-a) and 2-b) that $\displaystyle\lim_{n\to\infty}\int_{\partial C_n} \frac{cotan\pi w}{w-z} dw = 0$ (Hint : we can write $\dfrac{1}{w-z} = \dfrac{1}{w} + \dfrac{z}{w^2 - wz}$).

4) Deduce from 3-b) that $\displaystyle\frac{sin\pi z}{\pi z} = \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$.

<u>Exercise 5.</u>(1 hour)

Let $\gamma : [a, b] \to \mathbb{C}^*$ be a closed curve of class C^1.

1) Prove that for all $n \in \mathbb{Z}\backslash\{-1\}$, $\displaystyle\int_\gamma z^n dz = 0$.

Now we assume that $Im(\gamma) \subset D(0, r))$, $0 < r < 1$.

2) Prove that $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{n+1}(1-z)} = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$.

3) Let f be a non-constant holomorphic function on $D(0, r)$.

a) Prove that there exists s, $0 < s < r$ such that f has no zero on the circle of center 0 and radius s.

b) Prove that $I_s = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_s} \dfrac{f'(z)}{f(z)}dz$ is equal to the number of zeros of f counted with multiplicity in the disc $D(0,s)$, with $\gamma_s(t) = se^{it},\ t \in [0, 2\pi]$.

c) Consider $\Gamma_s = f \circ \gamma_s(t)$, $t \in [0, 2\pi]$. Prove that $Ind(\Gamma_s, 0) = I_s$. Deduce that there exists an open set $U \ni 0$ such that $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_s} \dfrac{f'(z)}{f(z) - a}dz = I_s$ for all $z \in U$.

d) Deduce the number of solutions of the equation $f(z) - a = 0$ in the disc $D(0,s)$.