KING SAUD UNIVERSITY
College of Sciences
Department of Mathematics

Final Examination / Semester II/ 1427/1428

Math 585

Question 1. Let $\Omega = \{z \in \mathbb{C} : -1 < Re(z) < 1\}$. We define

$$I(z) = \int_0^\infty \frac{x^z}{x^2 + x + 1} dx.$$

1) Prove that I is well defined on Ω.
2) Prove that I is analytic on Ω.
3) Evaluate the integral $I(\alpha)$, if α is a real number in $(0, 1)$.

Question 2. For $a \in \mathbb{C}$ and $s > 0$, we consider the set $\mathcal{F}$ of family of analytic functions on a domain $\Omega \subset \mathbb{C}$ satisfying to $|f(z)-a| > s$ for all $z \in \Omega$ and for all $f \in \mathcal{F}$. We consider the family $\mathcal{G} = \{g : g(z) = \dfrac{1}{f(z) - a},\ f \in \mathcal{F}\}$.
1) State the definition of a normal family and prove that $\mathcal{G}$ is normal.
2) Deduce that for any sequence $(g_n)_n$ of $\mathcal{G}$, we can extract a subsequence that converges to a function g which either identically equal to zero or without zero on Ω
3) Deduce that $\mathcal{F}$ is a normal family.

Question 3. 1) Let $\Omega = \{z \in \mathbb{C} : |z| > 1 \text{ and } Im(z) > 0\}$ be an open set in $\mathbb{C}$ and $f(z) = z + \dfrac{1}{z}$.
a) Find the image of the circle $|z| = 1$ and of the intervals $(1, \infty)$ and $(-\infty. - 1)$ by the map f. Deduce the set $f(\Omega)$.
b) Prove that f defines a conformal mapping from Ω into its image.
2) Describe the form of all mobious transformations that tranform the half plane $\{z \in \mathbb{C} : \mathcal{R}e(z) > 0\}$ into the unit disc.

Question 4. let $(a_n)_{n\geq 1}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} \dfrac{1}{|a_n|^2} < \infty$. Prove that the function

$$f(z) = \prod_{n=1}^{\infty} (1 - \frac{z}{a_n}) e^{\frac{z}{a_n}}$$

is holomorphic on $\mathbb{C}$.

Question 5. Let f be a non-constant holomorphic function on $D(0, r)$.

1) Prove that there exists s, $0 < s < r$ such that f has no zero on the circle of center 0 and radius s.

2) Prove that $I_s = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_s} \frac{f'(z)}{f(z)} dz$ is equal to the number of zeros of f counted with multiplicity in the disc $D(0, s)$, with $\gamma_s(t) = se^{it}$, $t \in [0, 2\pi]$.

3) Consider $\Gamma_s = f \circ \gamma_s(t)$, $t \in [0, 2\pi]$. Prove that $\mathrm{Ind}(\Gamma_s, 0) = I_s$ (Ind denotes the function index). Deduce that there exists an open set $U \ni 0$ such that

$$\frac{1}{2\pi i} \int_{\gamma_s} \frac{f'(z)}{f(z) - a} dz = I_s$$

for all $a \in U$.

4) For $a \in U$ deduce the number of solutions of the equation $f(z) - a = 0$ in the disc $D(0, s)$.