King Saud University
College of Sciences
Department of Mathematics

Semester-I/ 1428/1429
Time: 3 hours

Final Examination
Complex Analysis I (M-585)

Exercise 1.

Let $\{f_n : \Omega \to D\}_n$ be a sequence of holomorphic functions from a domain $\Omega \subset \mathbb{C}$ into a bounded domain $D \subset \mathbb{C}$. Assume that there exists a point $p \in \Omega$ such that $\{f_n(p)\}_n$ converges to a boundary point q of D. Prove that there exists a subsequence of $\{f_n\}_n$ that converges uniformly on all compacts of Ω to the boundary point q.
(Hint : consider the sequence $g_n(z) = f_n(z) - q$).

Exercise 2.

Let $\mathcal{P} = \{z \in \mathbb{C} : \mathcal{I}mz > 0\}$ and D be the unit disc in $\mathbb{C}$.
1) Let $a \in \mathcal{P}$. Prove that the function f_a defined as $f_a(z) = \dfrac{z-a}{z-\bar{a}}$ is a conformal transformation from $\mathcal{P}$ into D. Deduce the form of all conformal transformations from $\mathcal{P}$ into the unit disc.
2) let $f(z) = \dfrac{1}{1+z}$, $L = \{z \in \mathbb{C} : \mathcal{I}mz = 0\}$ and $\mathcal{C}$ be the unit circle. Determine $f(L)$ and $f(\mathcal{C})$. Deduce $f(\mathcal{P})$ and $f(D)$.
3) Let D be a domain in $\mathbb{C}$ and $\mathcal{F} = \{f \in H(D) : \mathcal{I}m(f) > 0\}$. Prove that $\mathcal{F}$ is a normal family (Hint : use question 1).

Exercise 3.

1) Let $k \in \mathbb{C}$, $|k| < 1$. Prove that the infinite product $g(z) = \displaystyle\prod_{n=1}^{\infty}(1 - \frac{z}{k^{-n}})$ defines an entire function. Verify that $g(kz) = (1-z)g(z)$.

2) Let f be an entire function non-identically equal to zero. Prove the set of zeros of f is countable. We denote this set by $Z(f) = \{a_1, a_2, ..., a_n, ..\}$.

a) If $Z(f)$ is finite, prove that there exist a polynomial function P and an entire function g such that $f(z) = P(z)e^{g(z)}$.

Now, assume that $Z(f)$ is infinite. After ordering the sequence $|a_n|$, we may assume that $\{|a_n|\}_n$ is an increasing sequence.

b) Prove that $|a_n| \to \infty$ as $n \to \infty$.

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c) Assume that $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$ is convergent. Prove that there exist entire functions g and h such that $f(z) = h(z)e^{g(z)}$.

Exercise 4.

Let D be the unit disc. We denote by $D^* = D \backslash \{0\}$.

1) Let f be an automorphism of D^*. Justify that 0 is a removable singularity. Deduce that f extends as an automorphism of D. If $\tilde{f}$ denotes this extension, prove that $\tilde{f}(0) = 0$.

2) Describe the analytic automorphism group of D^*.

Exercise 5.

1) By using an analytic argument, show that $\mathbb{C} \backslash \{0\}$ is not simply connected.

2) Let D be a simply connected domain in $\mathbb{C}$ and U be a harmonic function on D. Prove that there exists an analytic function f on D satisfying $U = \mathcal{R}e(f)$ on D.

3) Let U_1 and U_2 be harmonic functions on a domain $\Omega \subset \mathbb{C}$ such that $U_1^2 + U_2^2$ is harmonic on Ω. Prove that U_1 and U_2 are constant on Ω.

Exercise 6.

Evaluate the integral $\int_0^{2\pi} \frac{1}{a + cost}$, $a > 1$.