King Saud University
College of Sciences
Department of Mathematics

I-Semester 1428-1429
Time: 3 hours

Final Examination
Complex Analysis II (683)

Problem.

A

1) Prove that if D is a bounded domain in $\mathbb{C}^n$, then $Aut(D)$ is non-compact if and only if there exist a sequence $\{f_n\}$ of automorphism of D and a point $p \in D$ such that $\{f_n(p)\}$ converges to a boundary point q of D.

2) Assume that q is a strongly pseudonconvex boundary point of D and assume that for some point $p \in D$, the sequence $\{f_n(p)\}$ converge to q. Prove that after taking a subsequence of $\{f_n\}$ we may assume that $\{f_n\}$ converges uniformly on all compact of D to q.

B

Let $\Omega = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}$ and $\mathbb{B}$ be the unit ball in $\mathbb{C}^2$.

1) Verify that Ω is a pseudoconvex domain in $\mathbb{C}^2$, but not strongly pseudoconvex.

2) a) We denote by $\omega(\partial\Omega)$ the weakly pseudoconvex part of the boundary. Describe $\omega(\partial\Omega)$.

b) We denote by $Aut_0(\Omega) = \{f \in Aut(\Omega) \text{ and } f(0) = 0\}$. Justify that any automorphism $f \in Aut_0(\Omega)$ has the form :

$$f(z, w) = (az + cw, bz + dw)$$

where a, b, c and d are in $\mathbb{C}$ with $ad - bc \neq 0$.

c) Prove that if $f \in Aut_0(\Omega)$ then $f(\omega(\partial\Omega)) = \omega(\partial\Omega)$.

d) Deduce from c) that $|a| = 1$ and $b = 0$.

e)* Prove that $c = 0$ and $|d| = 1$.

f) Deduce the form of $Aut_0(\Omega)$ and justify that $Aut_0(\Omega)$ is compact.

g) Prove that there is no biholomorphic map between Ω and $\mathbb{B}$.

h) Give a proper holomorphic map from Ω to $\mathbb{B}$.

C

1) For $a \in \mathbb{C}$, $|a| < 1$, we define ψ_a as $\psi_a(z, w) = \left(\dfrac{a - z}{1 - \bar{a}z}, \dfrac{(1 - |a|)^{\frac{1}{4}}}{(1 - \bar{a}z)^{\frac{1}{2}}} w \right)$. Prove that ψ_a is an automorphism of Ω satisfying $\psi_a(a, 0) = (0, 0)$ and $\psi_a(0, 0) = (a, 0)$.

2) Deduce that $Aut(\Omega)$ is non compact.

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3) We admit that any automorphism $f \in Aut(\Omega)$ satisfies $f(\{(z,w) \in \Omega, w = 0\}) = \{(z,w) \in \Omega, w = 0\}$. Deduce the form of $Aut(\Omega)$.

Exercise 1.

1) State the definition of a domain of holomorphy.

2) Prove that any holomorphic maps on $\Delta^2(0,2) \backslash \overline{\Delta^2(0,1)}$ extends holomorphically to $\Delta^2(0,2)$.

3) Prove that $\Delta^2(0,2) \backslash \overline{\Delta^2(0,1)}$ is a not a pseudoconvex domain. Justify your answer.

4) Give an example of a pseudoconvex domain in $\mathbb{C}^2$ which is not convex.

5) Give an idea on the proof, showing that a domain of holomorphy is a pseudoconvex domain.

6) Justify that any domain in $\mathbb{C}$ is a domain of holomorphy.