King Saud University
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Department of Mathematics

Complex Analysis
1426-1427

Exercises S.2

Exercise 1. Precise the type of singularities of the following functions.
$\dfrac{1-cosz}{sin^2z}$, $z(e^{\frac{1}{z}})$, $z^2 sin\frac{z}{z+1}$, $sin(e^{\frac{1}{z}})$, $e^{cotan\frac{\pi}{z}}$.

Exercise 2. 1) Let f be a non-constant holomorphic function in an open set $\Omega \subset \mathbb{C}$ and let K be a compact of Ω. Prove that Ref can not have a maximum or a minimum in K (consider e^f).

2) Prove that if Ω is connected and f takes real values in the circle $\{z; |z-z_o| = R\} \subset \Omega$, then f is constant.

Exercise 3. Let $a \in \mathbb{C}$, $r > 0$ and f a holomorphic function in $D^*(a, r)$.

1) a) Prove that if a is an essential singularity , then there exists no neighborhood V of a such that $f(V\backslash\{a\}) \subset \{z \in \mathbb{C}, Rez > 0\}$.

b) Prove that if a is a pole of order $k \geq 1$, then we may write $f(z) = \dfrac{c}{(z-a)^k}(1+g(z))$, where c is constant and g is a holomorphic function and $g(a) = 0$. Deduce that any neighborhood of a intersects the set $\{Ref(z) < 0\}$.

c) Deduce that if there exists a neighborhood of a such that $f(V\backslash\{a\}) \subset \{z \in \mathbb{C}, Rez > 0\}$ then f extends in a.

Exercise 4. For $r > 0$, we denote by $D_r = \{z \in \mathbb{C} : |z| < r\}$. Let g be an analytic function in $\mathbb{C}$ such that $g(0) = 0$ and

$$\forall\ z \in \mathbb{C},\ \mathcal{R}eg(z) \leq a + |z|^\alpha$$

with $\alpha \in (0, 1)$ and $a \in \mathbb{R}$. For $r > 0$, we define the function

$$h(z) = \frac{g(rz)}{2A - g(rz)},\ \text{for } z \in D_1.$$

with $A = a + r^\alpha$.

a) Prove that h is analytic in D_1 and $h(D_1) \subset D_1$.

b) Deduce that $|g(rz)| \leq \dfrac{2A|z|}{1-|z|}$ $\forall\ z \in D_1$ and $|g(z)| \leq \dfrac{2A|z|}{r-|z|}$ $\forall\ z \in D_r$.

c) Deduce that g identically equal to zero in $\mathbb{C}$.

Exercise 5. Let $f : D(0, 1) \to D(0, 1)$ be an analytic function. Assume that f has two fixed points in $D(0, 1)$. Prove that $f = Id$ (we can use the Schwarz's lemma).

Exercise 6. Let $\Omega = \{z \in \mathbb{C} : -\pi < Imz < \pi\}$ and D the unit disc in $\mathbb{C}$. We consider $f(z) = \dfrac{1-z}{1+z}$ for $z \in D$.

1) Prove that f is analytic in D and find $f(D)$.

2) Deduce that $g = Log_{-\pi}(f^2)$ is well define and $g(0) = 0$.

3) Prove that g is a biholomorphism from D into Ω.

4) For $0 < r < 1$, we denote by D_r the disc of center 0 and radius r.

a) Prove that $f(D_r)$ is the disc of diameter $[\frac{1-r}{1+r}, \frac{1+r}{1-r}]$.

b) Deduce that $|Argf(z)| \leq 2Arctanr$ for $z \in \overline{D}_r$.

5) Let k be an analytic function in D such that $K(D)\Omega$ and $k(0) = 0$. Prove that $k(\overline{D}_r) \subset \overline{D}_r$ for $0 < r < 1$ (apply the Schwarz's lemma to $g^{-1} \circ k$)

6) Deduce that $Rek(z) \leq 2ln\dfrac{1+|z|}{1-|z|}$ and $|Imk(z)| \leq 4Arctan|z|$.