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SECOND MID-TERM EXAMINATION
Math-585

EXERCISE I.

Let $\Omega = \{z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Im}z > 0\}$ and $f(z) = \dfrac{1+z}{1-z}$.

a) Prove that $f(\{z : |z| = 1 \text{ and } \mathrm{Im}z > 0\}) = i\mathbb{R}^+$ (Hint : verify that $f(e^{i\theta}) = icot(\frac{\theta}{2})$).

b) Determine $f([-1, 1])$.

c) Determine $f(\frac{i}{2})$. Deduce that $f(\Omega) = D = \{z : Imz > 0 \text{ and } Rez > 0\}$ and that f defines a conformal transformation from Ω into the D.

EXERCISE II.

1- State Montel's theorem.

2- a) Let $\{f_n\}$ be a sequence of holomorphic functions on a domain $D \subset \mathbb{C}$. Suppose that $\{f_n\}$ is without zero on D and $\{f_n\}$ converges uniformly on all compacts of D to a holomorphic function f. Prove that f is either not identically equal to zero on D or without zero on D.

b) Application: Let $f_n : D \to \Delta(0, 1)$ be a sequence of holomorphic mappings from a domain D in $\mathbb{C}$ into the unit disc. Assume that for a certain point $p \in D$, the sequence $\{f_n(p)\}$ converges to a boundary point q, $|q| = 1$. Prove that (after taking a subsequence) $\{f_n\}$ converges uniformly on all compacts of D to q.
(Hint : consider $\varphi_n = f_n - q$).

EXERCISE III.

We denote by log the branch of the function logarithm defined on $\mathbb{C} \setminus i\mathbb{R}_-$ (i.e., $argz \in (-\frac{\pi}{2}, \frac{3\pi}{2})$).

1- Let $f(z) = \dfrac{logz}{1+z^2}$, $z \in \mathbb{C} \setminus i\mathbb{R}_-$.
Determine the poles of f and their corresponding residues.

2- Deduce the value of $\displaystyle\int_{\Gamma_{r,R}} f(z)dz$, where $\Gamma_{r,R} = [-R, -r] \vee C_r \vee [r, R] \vee C_R$, (see fig.1)

3- Prove that the following integrals are convergent and compute their values :

$$\int_0^\infty \frac{logx}{x^2+1}dx \quad \text{and} \quad \int_0^\infty \frac{dx}{x^2+1}.$$

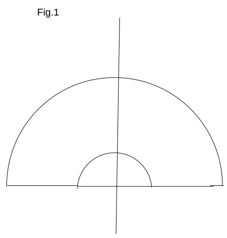

Fig. 1.

EXERCISE IV.

State Rouché 's theorem. Prove that if f is a holomorphic function in a neighborhood of the unit disc such $|f| < 1$ on the unit circle, then there exists only one point z_0 in the unit disc satisfying $f(z_0) = z_0$.