King Saud University
College of Sciences
Department of Mathematics

I-Semester 1427-1428
Time: 2 hours

Mid-Term Examination
Complex Analysis

Exercise 1.

For $R > 2$, let γ_R be the half-circle defined by $\gamma_R(t) = Re^{it}$, $t \in [0, \pi]$. We consider the function $f(z) = \frac{ze^{i2z}}{(z^2+4)^2}$.

a) Prove that the integral $\int_{\gamma_R} f(z)dz \to 0$ when $R \to +\infty$.

b) Precise the poles of the function f and compute their correspondent residues.

b) Use residue theorem to find the value of the integral

$$\int_0^{+\infty} \frac{x\sin 2x}{(x^2+4)^2} dx$$

Exercise 2.

Let $f : \mathbb{C} \to \mathbb{C}$ be a bijective analytic function (an automorphism of $\mathbb{C}$).

1) a) State the result of Liouville's theorem for analytic function.

b) Prove that $\lim_{|z|\to\infty} |f(z)| = \infty$ (Hint: we can use Liouville's theorem to get a contradiction).

2) Deduce that 0 is a pole for the function $g(z) = f(\frac{1}{z})$. Justify the existence of a positive integer m such that $g(z) = \sum_{j=-m}^{\infty} a_j z^j$.

3) Deduce that f is a polynomial function (prove that $a_j = 0$ for all $j \geq 1$).

4) Prove that there exist $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ such that $f(z) = az + b$ for any $z \in \mathbb{C}$ (Hint : Notice that f is bijective).

Date: 9 December 2006.