King Saud University
College of Sciences
Department of Mathematics

I-Semester 1427-1428
Time: 3 hours

Final Examination
Math 585

Exercise 1.(35 mn)

1) Precise the image of the half-plane $\{z \in \mathbb{C} : Rez > 0\}$ by the mapping $f(z) = z^2$ and give a conformal transformation from $\mathbb{C}\backslash\mathbb{R}$ into the unit disc.

2) If Ω is a simply connected domain in $\mathbb{C}$, different from $\mathbb{C}$, justify the non-existence of a conformal transformation from $\mathbb{C}$ into Ω.

3) Prove that any conformal self-transformation of $\mathbb{C}$ that fixes the origin is linear.

4) Let $a, b, c \in D(0,1)$ Prove that the function

$$f(z) = z(\frac{z-a}{1-\bar{a}z})^n(\frac{z-b}{1-\bar{b}z})^m - c$$

has exactly $n+m+1$ roots in $D(0,1)$ (use Rouché theorem).

Exercise 2.(25 mn)

1) State the maximum principle.
Let f be a non-constant holomorphic function in an open set $\Omega \subset \mathbb{C}$ and let K be a compact of Ω. Prove that $\mathcal{R}ef$ can not have a maximum or a minimum in K.

2) Let $f(z) = \prod_{n=1}^{\infty}(cos\frac{z}{n})$. Prove that f is holomorphic on $\mathbb{C}$.

3) State the Cauchy inequality and find all holomorphic mappings $f : \mathbb{C} \to \mathbb{C}$ that satisfie $|f(z)| \leq |z|$ (Hint: note that $f(0) = 0$).

Exercise 3.(1 hour)

We denote by D the unit disc. For $a, b \in D$, we define $\delta(a,b) = |\frac{b-a}{1-\bar{a}b}|$.

1) Prove that $\delta(a,b) < 1$ for all $a, b \in D$ (Hint: apply the maximum principle to the function $b \to \frac{b-a}{1-\bar{a}b}$.

Date: 5 Feb 2007.

2) Let $f : D \to D$ be a holomorphic function. We denote by $h(z) = \dfrac{z-a}{1-\bar{a}z}$, $k(z) = \dfrac{z - f(a)}{1 - \overline{f(a)}\ z}$ and $g = k \circ f \circ h^{-1}$.

a) Verify that g satisfies the hypothesis of Schwarz lemma.

b) Apply the Schwarz lemma to the function g to deduce that $\delta(f(a), f(b)) \leq \delta(a, b)$ for all $a, b \in D$.

c) Deduce from b) that if moreover f is bijective then $\delta(f(a), f(b)) = \delta(a, b)$ for all $a, b \in D$.

3) Conversely: assume that there exist $a \neq b$ in D such that $\delta(f(a), f(b)) = \delta(a, b)$. Prove that f is bijective from D to D (Hint: use the Schwarz lemma).

Exercise 4.(1 hour)

Let $f(z) = cot(\pi z)$.

1) Find the poles of the function f and their correspondent residues .

2) a) Let C_n the square contour with corners at $\pm(n+\frac{1}{2}) \pm i(n+\frac{1}{2})$ $n \in \mathbb{N}$. Prove that the function f is bounded on ∂C_n (the boundary of C_n) by a constant independent of n.

b) Use the Residues theorem to prove that for all $z \in \mathbb{C} \backslash \mathbb{Z}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{1}{2\pi i} \int_{\partial C_n} \frac{cot \pi w}{w - z} dw = cot \pi z - \frac{1}{\pi} \sum_{p=-n}^{p=n} \frac{1}{z - p}.$$

3) a) Verify that $\displaystyle\int_{\partial C_n^+} \frac{cot\pi w}{w} dw = -\int_{\partial C_n^-} \frac{cot \pi w}{w} dw$, where ∂C_n^+ denotes the vertical sides of the square and ∂C_n^- denotes the horizontal sides. Deduce that

$$\int_{\partial C_n} \frac{cot \pi w}{w} dw = 0.$$

b) Deduce from 3-a) and 2-b) that $\displaystyle\lim_{n\to\infty} \int_{\partial C_n} \frac{cot\pi w}{w - z} dw = 0$ (Hint : we can write $\dfrac{1}{w-z} = \dfrac{1}{w} + \dfrac{z}{w^2 - wz}$).

4) Deduce from 3-b) that $\dfrac{sin \pi z}{\pi z} = \displaystyle\prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$.