Complex Analysis I

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References

H.A. Priestly. it Introduction to Complex Analysis.
J. B Conway, Functions of one complex variables

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Exercises S.2
Presented by Dr. Nabil Ourimi

Exercise 1. Determine the type of singularities of the following functions.
$\dfrac{1-cosz}{sin^2 z}$, $z(e^{\frac{1}{z}})$, $z^2 sin\frac{z}{z+1}$, $sin(e^{\frac{1}{z}})$, $e^{cotan\frac{\pi}{z}}$.

Exercise 2. 1) Let f be a non-constant holomorphic function in an open set $\Omega \subset \mathbb{C}$ and let K be a compact of Ω. Prove that Ref can not have a maximum or a minimum in K (consider e^f).

2) Prove that if Ω is connected and f takes real values on the circle $\{z; |z-z_o| = R\} \subset \Omega$, then f is constant.

Exercise 3. Let $a \in \mathbb{C}$, $r > 0$ and f a holomorphic function in $D^*(a,r)$.

1) a) Prove that if a is an essential singularity , then there exists no neighborhood V of a such that $f(V \backslash \{a\}) \subset \{z \in \mathbb{C}, Rez > 0\}$.

b) Prove that if a is a pole of order $k \geq 1$, then we may write $f(z) = \dfrac{c}{(z-a)^k}(1+g(z))$, where c is constant and g is a holomorphic function and $g(a) = 0$. Deduce that any neighborhood of a intersects the set $\{Ref(z) < 0\}$.

c) Deduce that if there exists a neighborhood of a such that $f(V \backslash \{a\}) \subset \{z \in \mathbb{C}, Rez > 0\}$ then f extends in a.

Exercise 4. For $r > 0$, we denote by $D_r = \{z \in \mathbb{C} : |z| < r\}$. Let g be an analytic function in $\mathbb{C}$ such that $g(0) = 0$ and

$$\forall\ z \in \mathbb{C},\ \mathcal{R}eg(z) \leq a + |z|^\alpha$$

with $\alpha \in (0,1)$ and $a \in \mathbb{R}$. For $r > 0$, we define the function

$$h(z) = \frac{g(rz)}{2A - g(rz)},\ \text{for } z \in D_1.$$

with $A = a + r^\alpha$.

a) Prove that h is analytic in D_1 and $h(D_1) \subset D_1$.

b) Deduce that $|g(rz)| \leq \dfrac{2A|z|}{1-|z|}$ $\forall\, z \in D_1$ and $|g(z)| \leq \dfrac{2A|z|}{r-|z|}$ $\forall\, z \in D_r$.

c) Deduce that g identically equal to zero in $\mathbb{C}$.

Exercise 5. Let $f : D(0,1) \to D(0,1)$ be an analytic function. Assume that f has two fixed points in $D(0,1)$. Prove that $f = Id$ (we can use the Schwarz's lemma).

Exercise 6. Let $\Omega = \{z \in \mathbb{C} : -\pi < Imz < \pi\}$ and D the unit disc in $\mathbb{C}$. We consider $f(z) = \dfrac{1-z}{1+z}$ for $z \in D$.

1) Prove that f is analytic in D and find $f(D)$.

2) Deduce that $g = Log_{-\pi}(f^2)$ is well define and $g(0) = 0$.

3) Prove that g is a biholomorphism from D into Ω.

4) For $0 < r < 1$, we denote by D_r the disc of center 0 and radius r.

a) Prove that $f(D_r)$ is the disc of diameter $[\frac{1-r}{1+r}, \frac{1+r}{1-r}]$.

b) Deduce that $|Argf(z)| \leq 2Arctanr$ for $z \in \overline{D}_r$.

5) Let k be an analytic function in D such that $K(D)\Omega$ and $k(0) = 0$. Prove that $k(\overline{D}_r) \subset \overline{D}_r$ for $0 < r < 1$ (apply the Schwarz's lemma to $g^{-1} \circ k$)

6) Deduce that $Rek(z) \leq 2ln\dfrac{1+|z|}{1-|z|}$ and $|Imk(z)| \leq 4Arctan|z|$.

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Exercises S.3
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Exercise 1. Let $f(z) = 2z^5 + 6z - 1$

1) Prove that f has only one zero in the interval $[0, 1]$.
2) Prove that f has four zeros in $\{z \in \mathbb{C}, 1 < |z| < 2\}$.

Exercise 2. Let $\Omega = D(0, \frac{2}{3\sqrt{3}})$.

1) Prove that for all $z \in \Omega$, there exists only one $w = f(z) \in \mathbb{C}$, $|w| < \frac{1}{\sqrt{3}}$ and $w^3 + w - z = 0$.

2) Prove that for all $z \in \Omega$, $f(z) = \dfrac{1}{2i\pi} \displaystyle\int_{|\xi|=\frac{1}{\sqrt{3}}} \frac{\xi(3\xi^2+1)}{\xi^3 + \xi - z} d\xi$.

3) Deduce that f is analytic in Ω.

4) If $f(z) = \sum_{0 \leq n \leq \infty} a_n z^n$. Compute a_n.

Exercise 3.
1) For which α the following integral is convergent ?

$$I(\alpha) = \int_0^\infty \frac{x^\alpha}{1+x^2} dx$$

2) Evaluate $I(\alpha)$ for the values of α founded in question 1).

Exercise 4. Let $\Omega = \{z \in \mathbb{C}, -1 < Re(z) < 1\}$ and $a \in \Omega$. We define

$$I(a) = \int_0^\infty \frac{x^a}{x^2 + x + 1} dx.$$

1) Prove that I is well defined in Ω.

2) Prove that I is analytic in Ω.

3) Evaluate the integral $I(a)$ if a is a real number in $]0, 1[$.

Exercise 5. Let $f(z) = \pi cotan(\pi z)$.

1) Find the poles and the correspondent residus of the function f.

2) Let $g(z) = \dfrac{P(z)}{Q(z)}$ be a rationnal function with $degQ \geq degP+2$ and let $a_1, a_2, ..., a_m$ be the poles of g and $b_1, b_2, ..., b_m$ the correspondent residus. We assume that $a_q \notin \mathbb{N}$ for any $q \in \{1, ..., m\}$. We denote by γ_n the square contour with corners at $\pm(n+\frac{1}{2}) \pm i(n+\frac{1}{2})$ $n \in \mathbb{N}$.

1) Prove that there exist two constant M and K independent of n. such that

$$|\pi cotan(\pi z)| \leq M \quad \forall z \in \gamma_n$$

$$|g(z)| \leq \frac{K}{|z|^2} \quad \text{for} \quad |z| >> 1$$

2) Deduce that $\lim_{n\to\infty} \int_{\gamma_n} f(z)g(z)dz = 0$ and

$$\lim_{n\to\infty} \sum_{p=-n}^{n} g(p) = -\sum_{q=1}^{m} Res(f(z)g(z), a_q).$$

Application : Compute $\sum_{1}^{\infty} \dfrac{1}{a+bn^2}$, $a, b > 0$

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Exercises S.4
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EXERCISE 1.

1) If Ω is a simply connected domain in $\mathbb{C}$, justify the non-existence of a conformal transformation from $\mathbb{C}$ into Ω.

2) Precise the image of the half-plane $\{z \in \mathbb{C} : Rez > 0\}$ by the mapping $f(z) = z^2$ and give a conformal transformation from $\mathbb{C}\backslash\mathbb{R}$ into the unit disc.

3) Prove that any conformal self-transformation of $\mathbb{C}$ that fixes the origin is linear.

EXERCISE 2.

1) a) Prove that all mobious transformations that tranform the halp-plane of Poincaré $\{z : Im(z) > 0\}$ into it self have the form : $f(z) = \dfrac{az+b}{cz+d}$ with a, b, c, d reals numbers with $ad - bc > 0$.

b) Prove that all mobious transformation that tranform $\{z : Im(z) > 0\}$ into the unit disc have the form $f(z) = e^{i\theta}\dfrac{z-\alpha}{z-\bar{\alpha}}$

2) Find the image of the open set $\Omega = \{|z| > 1 \text{ and } Im(z) > 0\}$ by the map $f(z) = z + \frac{1}{z}$ and prove that it defines a conformal mapping from Ω into its image.

EXERCISE 3.

1) Prove that the Laplace operator in polar coordinates is given by

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

2)Prove that if U is harmonic in Ω then

$$\int_0^{2\pi} u(rcos\theta, rsin\theta)d\theta = Alnr + B \text{ for } r_o < r < r_1$$

A and B are constants independent of r and r_o and r_1 are such that $\{z : r_o < |z| < r_1\} \subset \Omega$.

3) Assume that u is harmonic in $D(0, R)\backslash\{0\}$ and bounded there. Prove that u extends as a harmonic mapping in $D(0, R)$.

EXERCISE 4.

Let φ be a function defined by $\varphi(e^{it}) = 1$ if $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $\varphi(e^{it}) = 0$ it $t \in]\frac{-\pi}{2}, \frac{\pi}{2}[$. We define the function f in the unit disc $D(0,1)$ by

$$f(z) = \frac{1}{2\pi} \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}} \frac{e^{it} + z}{e^{it} - z} dt.$$

a) Prove that f is holomorphic in the disc $D(0,1)$. We denote $U = Re(f)$.

b) Let $0 < \alpha < \beta < 2\pi$. Prove that $\lim_{r \to 1,\ t \to t_o} \frac{1}{2\pi} \int_{\alpha}^{\beta} P_r(\theta - t) d\theta = 0$ if $t_o \notin [\alpha, \beta]$, $P_r(\theta)$ denotes the Poisson kernel.

c) Deduce that $\lim_{r \to 1,\ t \to t_o} U(e^{it}) = \varphi(e^{it_o})$, for $t_o \in [0, 2\pi] \backslash \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

We denote by $\tilde{U}$ the extension of U in $\bar{D}(0,1) \backslash \{\pm i\}$.

d) Find $U(0)$ and prove that f is the only holomorphic function in $D(0,1)$ such that $U = Re(f)$ and $f(0) = \frac{1}{2}$

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Exercises S.5
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EXERCISE 1.

Let h be a holomorphic function injective on an open set Ω. Prove that that if $h(a) \neq 0$ for some $a \in \Omega$ then a is a simple zero for h.

Let h be a holomorphic function on the disc $D(0,r)$ injective on $D^* = D(0,r)\backslash\{0\}$. Prove that $h'(0) \neq 0$ and h is injective on the disc $D(0,r)$.

EXERCISE 2. Let g be a holomorphic and injective function on D^*. Let $z_o \in D^*$. We define the function $f(z) = \dfrac{1}{g(z) - g(z_o)}$. Let $r > 0$ such that $D(z_o, r) \subset D^*$. Prove that there exists $\alpha > 0$ such that for all $z \in D^*\backslash D(z_o, r)$: $|g(z) - g(z_o)| \geq \alpha$.

What is the type of singularity of 0. Deduce that either g extends as a holomorphic injective function in the disc $D(0,r)$, or 0 is a simple pole of g.

Find an example of such function.

EXERCISE 3.

Let D be a bounded domain in $\mathbb{C}$ that contains 0 and f be a holomorphic function on Ω such that $f(0) = 0$ and $f(D) \subset D$.

1) We denote by $f_1 = f$ and $f_n = f \circ f_{n-1}$.

a) Prove that the sequence $(f_n^{'}(0))_n$ is bounded.

b) Compute $f_n^{'}(0)$ as a function of $f^{'}(0)$ and deduce that $|f_n^{'}(0)| \leq 1$.

2) Assume that $f^{'}(0) = 1$. Prove that $f \equiv Id$. (Hint: if $f(z) = z + \sum_{p=n}^{+\infty} a_p z^p$ in a neighborhood of 0, prove that $f_k(z) = z + \sum_{p=n}^{+\infty} b_p z^p$ with $b_n = k a_n$ and deduce that $a_n = 0$).

3) Assume that there exists $k \geq 1$ such that ${f^{'}}^{(k)}(0) = 1$ (${f^{'}}^{(k)} = f' \circ f' \circ ... \circ f'$). Prove that f is an automorphism of D.

4) We denote $\mathcal{F} = \{f \in \mathcal{H}(D), f(D) \subset D \text{ and } f(0) = 0\}$.

a) Prove that $\mathcal{F}$ is closed in $\mathcal{H}(D)$.

b) Prove that $\mathcal{F}$ is a normal family in $\mathcal{F}$.

5) Assume D is simply connected and $|f'(0)| = 1$. Prove that f is an automorphism of D. (Hint : If g is a biholomorphism from D into the unit disc $D(0,1)$, consider the function $g \circ f \circ g^{-1}$).