Statistics 581, Problem Set 9 Solutions

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1. (a) Lehmann and Casella, problem 6.3.1, page 501: Let X have the binomial distribution $Bin(n, p)$, $0 \le p \le 1$. Determine the MLE of p:
(i) by the usual calculus method determining the maximum of a function.
(ii) by showing that $p^x q^{n-x} \le (x/n)^x [(n-x)/n]^{n-x}$.
(b) Lehmann and Casella, problem 6.3.2, page 501: In the preceding problem, show that the MLE does not exist when p is restricted to $0 < p < 1$ and when $X = 0$ or $X = 1$.
(c) Lehmann and Casella, problem 6.3.5, page 501: Here is a (slight) rephrasing of the problem: Let $X \sim$ Bernoulli(p): $P(X = 0) = 1 - p = q$, $P(X = 1) = p$. Suppose that it is known that $1/3 \le p \le 2/3$.
(i) Find the MLE;
(ii) Show that the expected squared error of the MLE is uniformly larger than that of $\delta(X) = 1/2$.

Solution: (a)(i) Since $\log P_p(X = x) = x \log p + (n-x)\log(1-p)$, we have $l(p|X) = X \log p + (n - X)\log(1-p)$; differentiating this with respect to p yields

$$\dot{l}(p|X) = \frac{X}{p} - \frac{n-X}{1-p} = \frac{X(1-p) - (n-X)p}{p(1-p)}$$

and this equals 0 if $p = \hat{p} \equiv X/n$. Since the second derivative is

$$\ddot{l}(p|X) = -\frac{X}{p^2} - \frac{n-X}{(1-p)^2} < 0$$

it follows that $\hat{p} = X/n$ is the MLE of $p \in [0, 1]$.
(a)(ii) Since $(\prod_{i=1}^n y_i)^{1/n} \le n^{-1}(y_1 + \cdots + y_n)$ for any numbers $y_i \ge 0$, it follows, with $y_i \equiv np/X$ for $i = 1, \ldots, X$, and $y_i \equiv nq/(n-X)$, $i = X+1, \ldots, n$, that

$$\left\{ \left(\frac{np}{X}\right)^X \left(\frac{nq}{n-X}\right)^{n-X} \right\}^{1/n} \le n^{-1} \left\{ X\frac{np}{X} + (n-X)\frac{nq}{n-X} \right\} = 1\,,$$

or, equivalently,

$$p^X (1-p)^{n-X} \le \left(\frac{X}{n}\right)^X \left(\frac{n-X}{n}\right)^{n-X}\,,$$

with equality if and only if $p = X/n \equiv \hat{p}$. Thus $\hat{p} = X/n$ is the MLE of $p \in [0, 1]$.
(b) When the closed interval $[0, 1]$ is replaced by the open interval $(0, 1)$, then the MLE exists if $0 < X < n$ and is $\hat{p} = X/n \in (0, 1)$ in this case. If $X = 0$, then the log-likelihood equals $n \log(1-p)$, so $\sup_{p \in (0,1)} l(p) = 0$, but this supremum is not achieved (in the set $(0, 1)$). Thus the MLE does not exist in this case. Similarly, if $X = n$, the the log-likelihood equals $n \log p$, so $\sup_{p \in (0,1)} l(p) = 0$, but this supremum is not achieved (in the set $(0, 1)$).

(c)(i) For the more general case in which $X \sim \text{Binomial}(n, p)$ From (a), the MLE $\hat{p}$ of $p \in [1/3, 2/3]$ is

$$\hat{p} = \begin{cases} X/n, & \text{if } X/n \in [1/3, 2/3], \\ 1/3, & \text{if } X/n < 1/3, \\ 2/3, & \text{if } X/n > 2/3\,. \end{cases}$$

For $n = 1$, this implies that the MLE is 1/3 if $X = 0$ and 2/3 if $X = 1$.
(ii) Now the estimator $\delta(X) = 1/2$ has expected squared error

$$R_1(p) \equiv E_p(\delta(X) - p)^2 = (1/2 - p)^2, \qquad 1/3 \le p \le 2/3\,.$$

On the other hand the MLE $\hat{p}$ has expected squared error

$$\begin{aligned} R_2(p) \equiv E_p(\hat{p} - p)^2 &= p(2/3 - p)^2 + q(1/3 - p)^2 \\ &> (1/2 - p)^2 = R_1(p) \end{aligned}$$

by noting that $R_2(1/3) = 1/3^3 > 1/6^2 = R_1(1/3)$, and $R_2(1/2) = (1/2)(1/6)^2 + (1/2)(1/6)^2 = 1/6^2 > 0 = R_1(1/2)$. See the following Figure 1 for a comparison of these two mean-square errors.

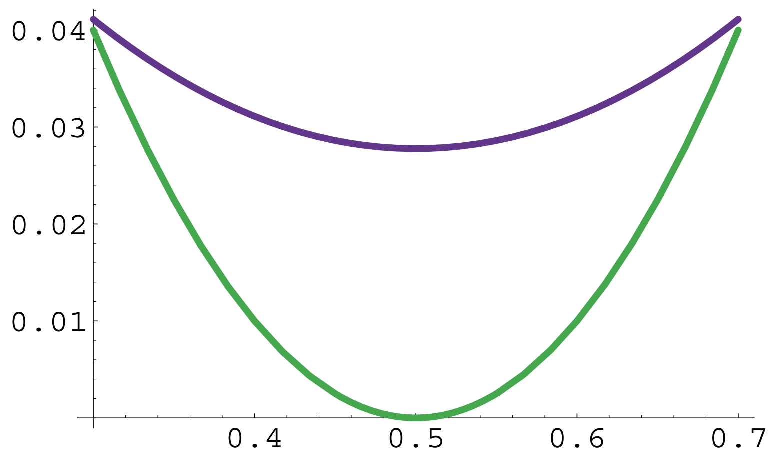

Figure 1: Mean-Square Errors.

(d) First consider problem 3.15(c): Here the log-likelihood for the scale family for a density f is

$$l_n(a) = n \log a + \sum_{i=1}^{n} \log f(aX_i), \qquad a > 0\,.$$

Hence the score equation is

$$\begin{aligned} 0 = \dot{l}_n(a) &= \frac{n}{a} + \sum_{i=1}^{n} \frac{f'(aX_i)}{f(aX_i)} X_i \\ &= \frac{1}{a}\left\{ n - \sum_{i=1}^{n} g(aX_i) \right\} \equiv \frac{1}{a}\{n - h_n(a)\} \end{aligned}$$

where $g(x) \equiv -xf'(x)/f(x)$. If $xf'(x)/f(x)$ is strictly decreasing in x, then $g(x)$ is strictly increasing in x, and hence $h_n(a)$ is strictly increasing in a. Hence there is at most one value of a satisfying $h_n(a) = n$. If a solution exists, it is unique.
For the particular case of a Cauchy density f, it is easy to compute

$$g(x) = \frac{2x^2}{1+x^2}$$

which is strictly increasing (from 0 at $x = 0$ to 2 at $x = \infty$). Moreover, in this case the likelihood equation becomes

$$h_n(a) = \sum_{i=1}^{n} \frac{2a^2X_i^2}{1+a^2X_i^2} = n\,.$$

But the left side converges to 0 as $a \downarrow 0$, and converges to $2n$ as $a \uparrow \infty$. Since it is monotone increasing and continuous, there is a unique solution $\hat{a}_n$. All the hypotheses of Theorem 1.5 hold in this case, and

$$I_{scale}(f) = \int_{-\infty}^{\infty} \left\{1 + x\frac{f'(x)}{f(x)}\right\}^2 f(x)dx = \frac{1}{2}\,.$$

Thus it follows that

$$\sqrt{n}(\hat{a}_n - a) \to_d N(0, 2a^2)\,.$$

2. Suppose that $(Y|Z) \sim$ Weibull$(\lambda^{-1}e^{-\gamma Z}, \beta)$, as in problem 2 of problem set # 8, where $Z \sim$ Bernoulli(η). Reparametrize as in problem set # 8 so that the joint density is $p_\theta(y,z)$ with $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ where $\theta_4 = \eta$. Thus the conditional hazard function is

$$\lambda_\theta(t|z) = \theta_3\theta_2 \exp(\theta_1 z)t^{\theta_3 - 1}\,,$$

$Z \sim$ Bernoulli(η), and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ with $\theta_4 \equiv \eta$. Let $X = (Y, Z)$, and suppose that we observe $X_1, \ldots, X_n$ i.i.d. as X.
(a) Find the score equations for estimation of θ.
(b) Give conditions on the data $X_1, \ldots, X_n = (Y_1, Z_1), \ldots, (Y_n, Z_n)$ guaranteeing that the score equations have a unique solution which maximizes the likelihood. Call the resulting estimators $\widehat{\theta}_n$.
(c) What does theorem 4.1.2 (Chapter 4, page 5), say about the asymptotic distribution of $\sqrt{n}(\widehat{\theta}_n - \theta_0)$ when the distribution of the data is given by P_{θ_0}.
(d) Suppose that $\theta_1 \neq \theta_0$ is the "true" value of the parameter θ, and we consider the likelihood ratio $L_n(\theta_1)/L_n(\theta_0)$ where $L_n(\theta) \equiv \prod_{i=1}^n p_\theta(X_i)$. Show that $n^{-1}\log(L_n(\theta_1)/L_n(\theta_0)) \to_p$ some constant, and identify the constant as explicitly as possible in terms of θ_1, θ_0.

Solution: (a) From problem set #8 we know that

$$\begin{aligned}
\dot{l}_1(Y,Z) &= Z(1 - \theta_2 e^{\theta_1 Z} Y^{\theta_3})\,,\\
\dot{l}_2(Y,Z) &= \frac{1}{\theta_2} - e^{\theta_1 Z} Y^{\theta_3}\,,\\
\dot{l}_3(Y,Z) &= \frac{1}{\theta_3} + \log Y(1 - \theta_2 e^{\theta_1 Z} Y^{\theta_3})\,,\\
\dot{l}_4(Y,Z) &= \frac{Z}{\eta} - \frac{1-Z}{1-\eta} = \frac{Z-\eta}{\eta(1-\eta)}\,.
\end{aligned}$$

Hence the score equations for $\theta = (\theta_1, \ldots, \theta_4)$ are

$$
\begin{aligned}
0 &= \sum_{i=1}^n \dot{\mathbf{l}}_1(Y_i, Z_i) = \sum_{i=1}^n Z_i(1 - \theta_2 e^{\theta_1 Z_i} Y_i^{\theta_3}), && (0.1)\\
0 &= \sum_{i=1}^n \dot{\mathbf{l}}_2(Y_i, Z_i) = \sum_{i=1}^n \left\{ \frac{1}{\theta_2} - e^{\theta_1 Z_i} Y_i^{\theta_3} \right\}, && (0.2)\\
0 &= \sum_{i=1}^n \dot{\mathbf{l}}_3(Y_i, Z_i) = \sum_{i=1}^n \left\{ \frac{1}{\theta_3} + \log Y_i (1 - \theta_2 e^{\theta_1 Z_i} Y_i^{\theta_3}) \right\}, && (0.3)\\
0 &= \sum_{i=1}^n \dot{\mathbf{l}}_4(Y_i, Z_i) = \sum_{i=1}^n \frac{Z_i - \eta}{\eta(1-\eta)}. && (0.4)
\end{aligned}
$$

(b) Let $\widehat{\theta} = (\widehat{\theta}_1, \ldots, \widehat{\theta}_4)$ denote the solution of this system of equations, assuming that it exists. It follows easily that

$$\widehat{\theta}_4 = \widehat{\eta} = \overline{Z}_n, \tag{0.5}$$

and, from (0.2),

$$\frac{1}{\widehat{\theta}_2} = \frac{1}{n} \sum_{i=1}^n e^{\widehat{\theta}_1 Z_i} Y_i^{\widehat{\theta}_3}. \tag{0.6}$$

Substitution of (0.6) into (0.1) and (0.3) yields

$$\sum_{i=1}^n Z_i \left(1 - \frac{e^{\widehat{\theta}_1 Z_i} Y_i^{\widehat{\theta}_3}}{n^{-1} \sum_{j=1}^n e^{\widehat{\theta}_1 Z_j} Y_j^{\widehat{\theta}_3}} \right) = 0, \tag{0.7}$$

and

$$\frac{n}{\widehat{\theta}_3} + \sum_{i=1}^n \log Y_i \left(1 - \frac{e^{\widehat{\theta}_1 Z_i} Y_i^{\widehat{\theta}_3}}{n^{-1} \sum_{j=1}^n e^{\widehat{\theta}_1 Z_j} Y_j^{\widehat{\theta}_3}} \right) = 0. \tag{0.8}$$

Thus we have reduced the original problem to two equations, (0.7) and (0.8) in the two parameters θ_1 and θ_3. Since each Z_i is either 0 or 1, the first of these two equations can be rewritten as

$$\sum_{i:Z_i=1} \left(1 - \frac{e^{\widehat{\theta}_1 Z_i} Y_i^{\widehat{\theta}_3}}{n^{-1} \sum_{j=1}^n e^{\widehat{\theta}_1 Z_j} Y_j^{\widehat{\theta}_3}} \right) = 0, \tag{0.9}$$

or equivalently

$$\sum_{i:Z_i=1} \left\{ e^{\widehat{\theta}_1} \sum_{j:Z_j=1} Y_j^{\widehat{\theta}_3} - n e^{\widehat{\theta}_1} Y_i^{\widehat{\theta}_3} + \sum_{j:Z_j=0} Y_j^{\widehat{\theta}_3} \right\} = 0, \tag{0.10}$$

or

$$
\begin{aligned}
e^{\widehat{\theta}_1} & \left\{ \left(\sum_1^n Z_i \right) \left(\sum_1^n Z_j Y_j^{\widehat{\theta}_3} \right) - n \sum_1^n Z_i Y_i^{\widehat{\theta}_3} \right\} \\
&+ \left(\sum_1^n Z_i \right) \left(\sum_1^n (1 - Z_i) Y_i^{\widehat{\theta}_3} \right) = 0.
\end{aligned}
$$

Assuming that $0 < \sum_1^n Z_i < n$ (i.e. not all $Z_i = 1$ and not all $Z_i = 0$), this yields

$$e^{\widehat{\theta}_1} = \frac{\sum_1^n Z_i \sum_1^n (1 - Z_i) Y_i^{\widehat{\theta}_3}}{(n - \sum_1^n Z_i) \sum_1^n Z_i Y_i^{\widehat{\theta}_3}} . \tag{0.11}$$

Note that the right side of (0.11) converges to 1 as $\widehat{\theta}_3 \to 0$, and as $\widehat{\theta}_3 \to \infty$ it converges to

$$\frac{(\sum_1^n Z_i)(1 - Z_{(n)})}{(n - \sum_1^n Z_i) Z_{(n)}} = \begin{cases} 0, & \text{if } Z_{(n)} = 1 \\ \infty, & \text{if } Z_{(n)} = 0 . \end{cases}$$

Here $Z_{(n)}$ is the Z_j corresponding to the largest Y_j, namely $Y_{(n)}$. On the other hand, for fixed values of θ_1, the third score equation (0.8) is

$$\frac{n}{\widehat{\theta}_3} + \sum_{i=1}^n \log Y_i \left(1 - \frac{e^{\theta_1 Z_i} Y_i^{\widehat{\theta}_3}}{n^{-1} \sum_{j=1}^n e^{\theta_1 Z_j} Y_j^{\widehat{\theta}_3}} \right) = 0 \tag{0.12}$$

or equivalently,

$$\frac{n}{\widehat{\theta}_3} + \sum_{i=1}^n \log Y_i \left(1 - \frac{e^{\theta_1 Z_i} Y_i^{\widehat{\theta}_3}}{n^{-1} \sum_{j=1}^n e^{\theta_1 Z_j} Y_j^{\widehat{\theta}_3}} \right) = 0 \tag{0.13}$$

or,

$$h(\widehat{\theta}_3, \theta_1) \equiv \frac{\sum_{i=1}^n [Z_i e^{\theta_1} + (1 - Z_i)] Y_i^{\widehat{\theta}_3} \log Y_i}{\sum_{j=1}^n [Z_j e^{\theta_1} + (1 - Z_j)] Y_j^{\widehat{\theta}_3}} - \frac{1}{\widehat{\theta}_3} = \frac{1}{n} \sum_1^n \log Y_i \tag{0.14}$$

where

$$h(v, w) \equiv \frac{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v \log Y_i}{\sum_{j=1}^n [Z_j e^w + (1 - Z_j)] Y_j^v} - \frac{1}{w} . \tag{0.15}$$

Now for each fixed w we have

$$\begin{aligned} \frac{\partial}{\partial v} h(v, w) &= \frac{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v (\log Y_i)^2}{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v} \\ &\quad - \left(\frac{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v \log Y_i}{\sum_{i=1}^n [Z_i e^w + (1 - Z_i)] Y_i^v} \right)^2 + \frac{1}{v^2} \\ &\equiv I + II > I \end{aligned}$$

and furthermore

$$I = \sum a_i^2 p_i - (\sum a_i p_i)^2 = Var_p(a) \tag{0.16}$$

with $a_i \equiv \log Y_i$,

$$p_i \equiv \frac{[Z_i e^w + (1 - Z_i)] Y_i^v}{\sum [Z_j e^w + (1 - Z_j)] Y_j^v} .$$

Thus $I > 0$ if not all the Y_i's are equal, and hence

$$\frac{\partial}{\partial v} h(v, w) > 0 . \tag{0.17}$$

Moreover

$$-\infty = \lim_{v \searrow 0} h(v,w) < \frac{\sum_{i=1}^n [Z_i e^w + (1-Z_i)] \log Y_i}{\sum_{i=1}^n [Z_i e^w + (1-Z_i)]} < \log Y_{(n)} = \lim_{v \to \infty} h(v,w)\,.$$

To see this last limit, note that with

$$p_{(i)} \equiv \frac{[Z_{(i)} e^w + (1-Z_{(i)})] Y_{(i)}^v}{\sum_{j=1}^n [Z_j e^w + (1-Z_j)] Y_j^v}$$

(here $Y_{(1)} \le \ldots \le Y_{(n)}$ are the ordered Y_i's and $Z_{(1)}, \ldots, Z_{(n)}$ are the corresponding Z's),

$$\begin{aligned} p_{(i)} &= \frac{[Z_{(i)} e^w + (1-Z_{(i)})]}{\sum_{j=1}^n [Z_{(j)} e^w + (1-Z_{(j)})](Y_{(j)}/Y_{(i)})^v} \\ &\to \begin{cases} 0, & i < n \ \ (\text{so} \ \ (Y_{(n)}/Y_{(i)}) > 1) \\ 1, & i = n \ \ (\text{so} \ \ Y_{(j)}/Y_{(n)} < 1, \ \ j < n) \end{cases} \end{aligned}$$

as $v \to \infty$. Thus (0.14) has a unique solution $\widehat{\theta}_3 = \widehat{\theta}_3(\theta_1)$ for each fixed θ_1. This is as far as I have succeeded in going with this problem. A summary of what I know is as follows: there is clearly trouble in estimating θ_1 if all the Z_i's are either 0 or 1; and monotonicity of the (condensed version of) the score equation for θ_3 will fail if all the Y_j's are equal. It seems likely that there is a unique solution when $0 < \sum_1^n Z_i < n$ and $Y_j \neq Y_k$ for some $j \neq k$, but I do not yet have a proof.

(c) Theorem 4.1.2 yields (after somewhat painful verification of the third derivative condition)

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \to_d N_4(0, I^{-1}(\theta_0))$$

where $I(\theta_0)$ was computed in problem 8.2.

(d) When $\theta_1 \neq \theta_0$ is true,

$$\begin{aligned} n^{-1} \log \frac{L_n(\theta_1)}{L_n(\theta_0)} &= n^{-1} \sum_{i=1}^n \log \frac{p_{\theta_1}(Y_i, Z_i)}{p_{\theta_0}(Y_i, Z_i)} \to_{a.s.} E_{\theta_1} \log \frac{p_{\theta_1}(Y_1, Z_1)}{p_{\theta_0}(Y_1, Z_1)} \\ &= K(P_{\theta_1}, P_{\theta_0}) \\ &= \log \frac{\theta_{12}\theta_{13}}{\theta_{02}\theta_{03}} + (\theta_{11} - \theta_{01})\theta_{14} + (\theta_{13} - \theta_{03}) E_{\theta_1} \log Y - E_{\theta_1} W_1 \\ &\qquad + E_{\theta_1} W_0 + \theta_{14} \log \frac{\theta_{14}}{\theta_{04}} + (1-\theta_{14}) \log \frac{1-\theta_{14}}{1-\theta_{04}}\,, \end{aligned}$$

where

$$\begin{aligned} W_1 &\equiv \theta_{12} e^{\theta_{11}} Z Y^{\theta_{13}}\,, \\ W_0 &\equiv \theta_{02} e^{\theta_{01}} Z Y^{\theta_{03}} \\ &= \frac{\theta_{02}}{\theta_{12}^{\theta_{03}/\theta_{13}}} \exp((\theta_{01} - \theta_{11}\theta_{03}/\theta_{13}) Z) W_1^{\theta_{03}/\theta_{13}}\,. \end{aligned}$$

so that

$$\begin{aligned} E_{\theta_1} W_1 &= 1 \\ E_{\theta_1} W_0 &= E_{\theta_1}(W_1^{\theta_{03}/\theta_{13}}) \frac{\theta_{02}}{\theta_{12}^{\theta_{03}/\theta_{13}}} E_{\theta_1} \exp((\theta_{01} - \theta_{11}\theta_{03}/\theta_{13})Z) \\ &= \Gamma(\theta_{03}/\theta_{13}+1) \frac{\theta_{02}}{\theta_{12}^{\theta_{03}/\theta_{13}}} \{\theta_{14}\exp(\theta_{01} - \theta_{11}\theta_{03}/\theta_{13}) + (1-\theta_{14})\} . \end{aligned}$$

It remains only to compute

$$\begin{aligned} E_{\theta_1} \log Y &= \frac{1}{\theta_{13}} E_{\theta_1} \log Y^{\theta_{13}} \\ &= \frac{1}{\theta_{12}\theta_{13}} E_{\theta_1}\{\exp(-\theta_{11}Z) E_{\theta_1}\{\theta_{12}\exp(\theta_{11}Z)\log Y^{\theta_{13}} | Z\} \\ &= \frac{1}{\theta_{12}\theta_{13}} E_{\theta_1}\{\exp(-\theta_{11}Z)\} E_{\theta_1}(W_1) \\ &= \frac{1}{\theta_{12}\theta_{13}} \{\theta_{14}\exp(-\theta_{11}) + (1-\theta_{14}\}\psi(1) \end{aligned}$$

where $\psi(1) = -\gamma$ and γ is Euler's constant.

3. Consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities

$$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1} \exp(-(\frac{x}{\alpha})^\beta) 1_{[0,\infty)}(x)$$

where $\theta \equiv (\alpha, \beta) \in (0,\infty) \times (0,\infty) \subset R^2$. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with density function p_θ.
A. If $X \sim P_\theta \in \mathcal{P}$, show that the distributions of $\log X$ form a location and scale family from a Gumbel (extreme value) density on R.
B. Use the result of A to construct method of moments estimators or quantile based estimators $\overline{\theta}_n$ of $\theta = (\alpha, \beta)$.
C. Show that the method of moments or quantile estimators $\overline{\theta}_n$ of θ are asymptotically normal, and find the asymptotic distribution; i.e. show that

$$\sqrt{n}(\overline{\theta}_n - \theta) \to_d N_2(0, \Sigma) \qquad \text{for some} \qquad \Sigma.$$

D. Does a maximum likelihood estimate of $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ exist? Is it unique?
E. Compute an approximate (one - step) maximum likelihood estimate $\check{\theta}$ of θ using the method of moment estimators $\overline{\theta}_n$ as the preliminary estimators based on the following data (with $n = 19$):

0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50,
7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89 .

[These are failure times in minutes for an insulating fluid between two electrodes subject to a voltage of 34 kV. – from Nelson, *Applied Life Data Analysis*, page 105.]

F. Compute the maximum likelihood estimator $\hat{\theta}_n$, and compare it with the one step estimator computed in E.

Solution: A. Recall that $Y \equiv (X/\alpha)^\beta \sim \exp(1)$, and that $W \equiv -\log(Y) \sim$ Gumbel:

$$P(W \leq w) = P(-\log(Y) \leq w) = P(Y \geq e^{-w}) = \exp(-e^{-w}).$$

Thus it follows that

$$W = -\log(Y) = \beta\{-\log(X) + \log(\alpha)\},$$

or equivalently that

$$T \equiv -\log(X) = \frac{1}{\beta}W - \log(\alpha).$$

Thus the distributions of $T \equiv -\log(X)$ form a location - scale family of the Gumbel (extreme value) distribution with d.f. $\exp(-\exp(-x))$.

B. Now $T = -\log X$ has

$$E(T) = \frac{\gamma}{\beta} - \log\alpha, \qquad Var(T) = \frac{1}{\beta^2}\frac{\pi^2}{6}$$

where $\gamma = .577...$ is Euler's constant (don't confuse this with the γ above!). Since $\overline{T} = -1.7864...$ and $S_T = 1.4853...$, moment estimators of (α, β) based on (8) are given by

$$\overline{\beta}_n \equiv \frac{\pi}{\sqrt{6}}\frac{1}{S_T} = .8639,$$

$$\overline{\alpha} = \exp(-\overline{T} + \frac{\gamma}{\overline{\beta}}) = 11.6407$$

for the given data.

C. Asymptotic normality of $(\overline{\alpha}_n, \overline{\beta}_n)$ follows from joint asymptotic normality of $(\overline{T}_n, S_T^2)$ and the delta method: by the multivariate CLT and Slutsky's theorem

$$\begin{pmatrix} \sqrt{n}(\overline{T} - ET)/\sigma \\ \sqrt{n}(S_T^2 - \sigma_T^2)/(\sqrt{2}\sigma_T^2) \end{pmatrix} \rightarrow_d \underline{Z} \sim N_2(0, \Sigma).$$

Then since $(\overline{\alpha}, \overline{\beta}) = g(\overline{T}, S_T^2)$ and $(\alpha, \beta) = g(E_\theta T, Var_\theta(T))$ where $g \equiv (g_1, g_2) : R^2 \rightarrow R^2$ is defined by

$$g_1(x, y) = \exp(\frac{\gamma\sqrt{6}}{\pi}\sqrt{y} - x),$$

$$g_2(x, y) = \frac{\pi/\sqrt{6}}{\sqrt{y}},$$

it follows by the delta method with $\underline{\tilde{Z}} \equiv (Z_1, \sqrt{2}\sigma_T^2 Z_2)$ that

$$\sqrt{n}\left((\overline{\alpha}_n, \overline{\beta}_n)^T - (\alpha, \beta)^T\right) \rightarrow_d \nabla g \underline{\tilde{Z}}$$

where

$$\nabla g \equiv \nabla g(E_\theta T, \mathrm{Var}_\theta T) = \begin{pmatrix} -\alpha & (3\gamma/\pi^2)\alpha\beta \\ 0 & -3\beta^3/\pi^2 \end{pmatrix}.$$

D. The maximum likelihood estimator exists and is unique in this model if not all the X_i's are equal (which happens with probability 1 if the model holds). The following solution is from Lehmann, TPE, page 536 (with slightly different notation).
We first reparametrize the Weibull model by writing

$$\begin{aligned} p_\theta(x) &= \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}\exp(-(\frac{x}{\alpha})^\beta)1_{(0,\infty)}(x) \\ &= \frac{\beta}{\eta}x^{\beta-1}\exp(-\frac{x^\beta}{\eta}) \\ &\equiv p_\gamma(x) \end{aligned}$$

where $\eta \equiv \alpha^\beta$ and $\gamma \equiv (\beta, \eta)$. Then

$$l(\gamma|\underline{X}) = n\log\beta - n\log\eta + (\beta-1)\sum_{i=1}^n \log X_i - \frac{1}{\eta}\sum_{i=1}^n X_i^\beta .$$

Thus, with $\gamma_1 \equiv \beta$, $\gamma_2 \equiv \eta$, the likelihood equations become

$$\dot{l}_1(\gamma|\underline{X}) = \frac{n}{\beta} + \sum_{i=1}^n \log X_i - \frac{1}{\eta}\sum_{i=1}^n X_i^\beta \log X_i = 0, \tag{0.18}$$

and

$$\dot{l}_2(\gamma|\underline{X}) = -\frac{n}{\eta} + \frac{1}{\eta^2}\sum_{i=1}^n X_i^\beta = 0, \tag{0.19}$$

or

$$\hat{\eta}_n = \frac{1}{n}\sum_{i=1}^n X_i^{\hat{\beta}} \tag{0.20}$$

from 0.19. Substitution of 0.20 into 0.18 yields the equation

$$\frac{\sum_i X_i^{\hat{\beta}}\log X_i}{\sum_i X_i^{\hat{\beta}}} - \frac{1}{\hat{\beta}} = \frac{1}{n}\sum_{i=1}^n \log X_i\,, \tag{0.21}$$

or

$$h(\hat{\beta}) = \frac{1}{n}\sum_{i=1}^n \log X_i \tag{0.22}$$

where

$$h(\beta) \equiv \frac{\sum_i X_i^\beta \log X_i}{\sum_i X_i^\beta} - \frac{1}{\beta} < \frac{\sum_i X_i^\beta \log X_i}{\sum_i X_i^\beta}$$

since $\beta > 0$. Now

$$\begin{aligned} h'(\beta) &= \frac{\sum_i X_i^\eta (\log X_i)^2}{\sum_i X_i^\beta} - (\frac{\sum_i X_i^\beta \log X_i}{\sum_i X_i^\beta})^2 + \frac{1}{\beta^2} \\ &\equiv I + II \\ &> I, \end{aligned}$$

and furthermore,

$$I = \sum a_i^2 p_i - (\sum a_i p_i)^2 = \mathrm{Var}_p(a)$$

since, with $a_i \equiv \log X_i$, $p_i \equiv X_i^\beta / \sum_j X_j^\beta \geq 0$, $\sum_i p_i = 1$. Thus $I > 0$ and hence $h'(\beta) > 0$ from (0.23) while

$$-\infty = \lim_{\beta \to 0} h(\beta) < \frac{1}{n} \sum_{i=1}^{n} \log X_i < \log X_{(n)} = \lim_{\beta \to \infty} h(\beta).$$

[Draw the picture!] (To see this last limit, note that with $p_{(i)} \equiv X_{(i)}^\beta / \sum_j X_j^\beta$,

$$\begin{aligned} p_{(i)} &= \frac{1}{(\frac{X_{(1)}}{X_{(i)}})^\beta + \ldots + (\frac{X_{(n)}}{X_{(i)}})^\beta} \\ &\to \begin{cases} 0, & i \leq n \quad \text{(so } X_{(n)}/X_{(i)} > 1) \\ 1, & i = n \quad \text{(so } X_{(j)}/X_{(n)} < 1,\ j < n) \end{cases} \end{aligned}$$

as $\beta \to \infty$.) Thus (0.22) has a unique solution $\hat{\beta}$. By taking this value of $\hat{\beta}$ in (0.20), we see that the MLE $\hat{\gamma}$ of γ exists and is unique. Thus the unique MLE of $\theta = (\alpha, \beta)$ is $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ with $\hat{\alpha} = \hat{\eta}^{1/\hat{\beta}}$.
E. The one step estimator using $\hat{I}(\overline{\theta}_n) = I(\overline{\theta}_n)$ is

$$\check{\theta}_n \equiv \overline{\theta}_n + \hat{I}_n^{-1}(\overline{\theta}_n)(\frac{1}{n}\dot{l}(\overline{\theta}_n)) = (12.27\ldots,\ 0.7421\ldots).$$

The one - step estimator using $\hat{I}_n(\overline{\theta}_n) = (-n^{-1}\ddot{l}_n(\overline{\theta}_n)$ is

$$\check{\theta}_n = (11.778, 0.7633).$$

F. The maximum likelihood estimate $\hat{\theta}_n = (12.222\ldots, 0.77082\ldots)$; see the following pages.

Mathematica input:

```
(* Here is the data:  *)
x = {.19, .78, .96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85,
  6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91,
  36.71, 72.89}
(* NSS is the sample size *)
NSS:= Length[x]
(* Some useful functions:  *)
(* f is the Weibull density function:  *)
f[t_,a_,b_] := (b/a)*(t/a)^(b-1) *Exp[-(t/a)^b] ;

(* aa and bb are the constants in the Weibull Informaton:  *)
aa := N[-(1-EulerGamma)];
bb := N[(Pi^2)/6 + aa^2 ]
(* Inf is the information matrix *)
Inf[a_,b_] := { {b^2/a^2 , aa/a}, {aa/a, bb/b^2}} ;
(* L  is the log-likelihood *)
L[a_,b_] := Sum[Log[f[x[[i]], a,b]], {i,1,NSS} ] ;
(* Sc is the vector of Scores *)
```

```
Sc[a_,b_] := Sum[ {(b/a)(((x[[i]]/a)^b) -1),
(1/b)(1-Log[(x[[i]]/a)^b]*((x[[i]]/a)^b  -1) ) },
{i,1,NSS}] ;
a[b_] := (Sum[x[[i]]^b, {i,1,NSS}]/NSS )^(1/b)

Plot3D[L[a,b], {a,4,15}, {b,.1,1.5}]
FindMinimum[-L[a,b], {a,10},  {b,2}]
ahat = 12.222
bhat = .770821
Wald[a_,b_] := NSS*({ahat,bhat} -{a,b}).Inf[ ahat,bhat].({ahat,bhat} -{a,b})
Rao[a_,b_] := Sc[a,b].Inverse[Inf[a,b]].Sc[a,b]/NSS
LR[a_,b_] := 2*(L[ahat,bhat] - L[a,b])
Wald[10,1]
Rao[10,1]
LR[10,1]
```

Mathematica output:

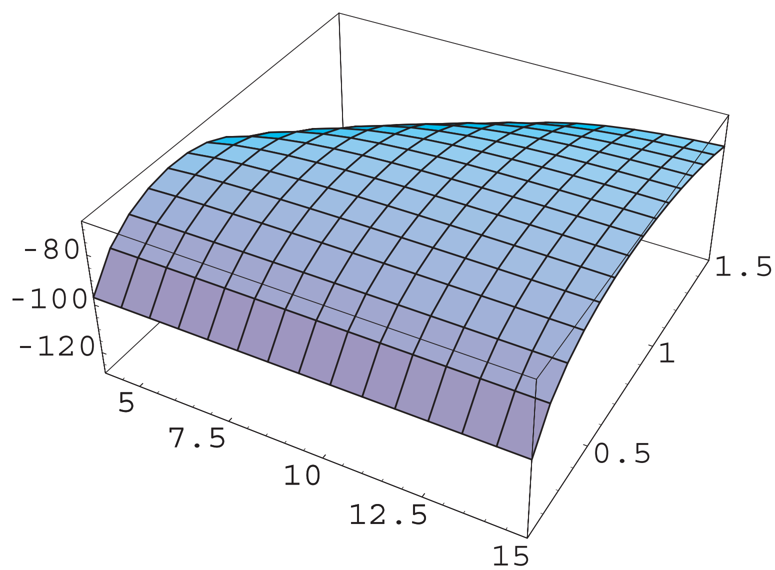

Figure 2: Weibull likelihood.

```
{68.386, {a -> 12.2222, b -> 0.770821}}
12.222
0.770821
4.10551
10.8385
5.29018
```