MA20033 - Solution Sheet Six

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1. **Suppose $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, \sigma^2 = 20)$ random quantities. Consider the hypothesis test**

$$H_0 : \mu = 100 \quad \text{versus} \quad H_1 : \mu = 95.$$

   (a) **Use the Neyman-Pearson lemma to find the critical region for a test with significance $\alpha = 0.05$. Your critical region should be expressed as simply as possible.**

If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ with σ^2 known, then the likelihood is given by

$$\begin{aligned} L(\mu) &= \prod_{i=1}^{n} f(x_i|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\} \\ &= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}. \end{aligned}$$

For $\sigma^2 = 20$ we have

$$\begin{aligned} L(100) &= (40\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{40}\sum_{i=1}^{n}(x_i - 100)^2\right\}, \\ L(95) &= (40\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{40}\sum_{i=1}^{n}(x_i - 95)^2\right\}. \end{aligned}$$

We consider $\lambda(x_1, \ldots, x_n; 100, 95) = L(100)/L(95)$. We have

$$\begin{aligned} \frac{L(100)}{L(95)} &= \exp\left[-\frac{1}{40}\sum_{i=1}^{n}\left\{(x_i - 100)^2 - (x_i - 95)^2\right\}\right] \\ &= \exp\left[-\frac{1}{40}\sum_{i=1}^{n}\left\{(100^2 - 95^2) - 2(100 - 95)x_i\right\}\right] \\ &= \exp\left\{-\frac{1}{40}(975n - 10n\overline{x})\right\}. \end{aligned}$$

The critical region given by the Neyman-Pearson lemma is

$$
\begin{aligned}
C^* &= \{(x_1, \ldots, x_n) : L(100)/L(95) \le k\} \\
&= \left\{(x_1, \ldots, x_n) : \exp\left\{-\frac{1}{40}(975n - 10n\overline{x})\right\} \le k\right\} \\
&= \left\{(x_1, \ldots, x_n) : \frac{1}{4}n\overline{x} - \frac{195}{8}n \le \log k\right\} \\
&= \{(x_1, \ldots, x_n) : \overline{x} \le c\}
\end{aligned}
$$

where c is the critical value chosen to ensure the test has significance $\alpha = 0.05$. Thus,

$$
\begin{aligned}
P(\overline{X} \le c | H_0 \text{ true}) &= P\{\overline{X} \le c | \overline{X} \sim N(100, 20/n)\} \\
&= P\left(Z \le \frac{c - 100}{\sqrt{20/n}}\right) = \Phi\left(\frac{c - 100}{\sqrt{20/n}}\right) = 0.05.
\end{aligned}
$$

Hence, from standard normal tables,

$$
\frac{c - 100}{\sqrt{20/n}} = -1.645
$$

so that $c = 100 - 1.645\sqrt{20/n}$ and the critical region of the 5% significance test with largest power is

$$
C^* = \{(x_1, \ldots, x_n) : \overline{x} \le 100 - 1.645\sqrt{20/n}\}.
$$

(b) **Find the corresponding probability of a Type II error (in terms of n and the Normal CDF Φ).**

Let $\beta = P(\text{Type II error})$. Then

$$
\begin{aligned}
\beta &= P(\text{Do not reject } H_0 | H_1 \text{ true}) \\
&= P\{\overline{X} > 100 - 1.645\sqrt{20/n} \mid \overline{X} \sim N(95, 20/n)\} \\
&= P(Z > 5\sqrt{n/20} - 1.645) = 1 - \Phi(5\sqrt{n/20} - 1.645).
\end{aligned}
$$

2. **Suppose $X_1, \ldots, X_{100}$ are independent and identically distributed $B(n = 10, p)$ random quantities. We wish to test the hypothesis**

$$
H_0 : p = 0.5 \quad \text{versus} \quad H_1 : p = 0.75.
$$

(a) **Use the Neyman-Pearson lemma to find the test of significance α with the largest power. Your critical region should be expressed as simply as possible.**

If $X_1, \ldots, X_{100}$ are iid $B(10, p)$ then the likelihood is given by

$$
\begin{aligned}
L(p) = \prod_{i=1}^{100} P(X_i = x_i | p) &= \prod_{i=1}^{100} \binom{10}{x_i} p^{x_i}(1-p)^{10-x_i} \\
&= \left\{\prod_{i=1}^{100} \binom{10}{x_i}\right\} p^{\sum_{i=1}^{100} x_i}(1-p)^{1000-\sum_{i=1}^{100} x_i}.
\end{aligned}
$$

Thus,

$$\begin{aligned}
L(0.5) &= \left\{\prod_{i=1}^{100}\binom{10}{x_i}\right\}(0.5)^{\sum_{i=1}^{100} x_i}(0.5)^{1000-\sum_{i=1}^{100} x_i}, \\
L(0.75) &= \left\{\prod_{i=1}^{100}\binom{10}{x_i}\right\}(0.75)^{\sum_{i=1}^{100} x_i}(0.25)^{1000-\sum_{i=1}^{100} x_i}.
\end{aligned}$$

We consider $\lambda(x_1, \ldots, x_{100}; 0.5, 0.75) = L(0.5)/L(0.75)$. We have

$$\frac{L(0.5)}{L(0.75)} = \frac{(0.5)^{\sum_{i=1}^{100} x_i}(0.5)^{1000-\sum_{i=1}^{100} x_i}}{(0.75)^{\sum_{i=1}^{100} x_i}(0.25)^{1000-\sum_{i=1}^{100} x_i}} = \left(\frac{2}{3}\right)^{\sum_{i=1}^{100} x_i} 2^{1000-\sum_{i=1}^{100}}.$$

Now, applying the Neyman-Pearson lemma, we construct the critical region by considering

$$\begin{aligned}
C^* &= \left\{(x_1, \ldots, x_{100}) : \left(\frac{2}{3}\right)^{\sum_{i=1}^{100} x_i} 2^{1000-\sum_{i=1}^{100} x_i} \leq k\right\} \\
&= \left\{(x_1, \ldots, x_{100}) : \left(\sum_{i=1}^{100} x_i\right)\log\frac{2}{3} + \left(1000 - \sum_{i=1}^{100} x_i\right)\log 2 \leq \log k\right\} \\
&= \left\{(x_1, \ldots, x_{100}) : \sum_{i=1}^{100} x_i \left(\log\frac{2}{3} - \log 2\right) \leq \log k - 1000\log 2\right\} \\
&= \left\{(x_1, \ldots, x_{100}) : \sum_{i=1}^{100} x_i \geq \frac{\log k - 1000\log 2}{\log 2/3 - \log 2}\right\} \\
&= \left\{(x_1, \ldots, x_{100}) : \sum_{i=1}^{100} x_i \geq c\right\}
\end{aligned}$$

where c is the constant chosen such that the significance of the test is $\alpha = 0.05$.

(b) **What is the sampling distribution of your test statistic? Hence find the critical value for a test of significance $\alpha = 0.05$ (you do not need to evaluate this critical value, but express it in terms of a quantile of the distribution of your test statistic).**

Since the $\{X_i\}$ are iid $B(10, p)$ random quantities, $\sum_{i=1}^{100} X_i \sim B(100 \times 10, p)$. Therefore, as c is chosen so that

$$P\left\{\sum_{i=1}^{100} X_i \geq c \;\middle|\; \sum_{i=1}^{100} X_i \sim B(1000, 0.5)\right\} = 0.05,$$

c is the 0.95-quantile of a $B(1000, p = 0.5)$ distribution. This quantile could be found computationally, or by using the Normal approximation to the Binomial distribution.

3. **Suppose $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu = 0, \sigma^2)$ random quantities and we wish to test the hypothesis**

$$H_0 : \sigma^2 = \sigma_0^2 \quad \text{versus} \quad H_1 : \sigma^2 = \sigma_1^2$$

where $\sigma_1^2 < \sigma_0^2$.

(a) **Use the Neyman-Pearson lemma to find the test of significance α which has the largest power. Try to ensure that your final test statistic is as simple as possible.**

If $X_1, \dots, X_n$ are iid $N(0, \sigma^2)$ then the likelihood is given by

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x_i^2}{2\sigma^2}\right) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right).$$

We consider $\lambda(x_1, \dots, x_n; \sigma_0^2, \sigma_1^2) = L(\sigma_0^2)/L(\sigma_1^2)$. We have

$$\frac{L(\sigma_0^2)}{L(\sigma_1^2)} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n} x_i^2\right\}.$$

The Neyman-Pearson lemma tells us that the critical region which maximises the power for a fixed probability of a Type I error is

$$\begin{aligned}
C^* &= \left\{(x_1, \dots, x_n) : \left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n} x_i^2\right\} \leq k\right\} \\
&= \left\{(x_1, \dots, x_n) : \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n} x_i^2\right\} \leq \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} k\right\} \\
&= \left\{(x_1, \dots, x_n) : \frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum_{i=1}^{n} x_i^2 \leq \log\left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} k\right\} \\
&= \left\{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i^2 \leq \frac{\log\left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} k}{\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)}\right\}.
\end{aligned}$$

In other words, we have a test of the form reject H_0 if $\sum_{i=1}^{n} x_i^2 \leq c$ where c is chosen so that the test has a significance of α.

(b) **What is the sampling distribution of your test statistic? Hence find the critical value for a test of significance $\alpha = 0.05$ (you do not need to evaluate this critical value, but express it in terms of a quantile of the distribution of your test statistic).**

Under H_0, $X_i \sim N(0, \sigma_0^2)$, $X_i/\sigma_0 \sim N(0,1)$, so $X_i^2/\sigma_0^2 \sim \chi_1^2$ and hence $\frac{1}{\sigma_0^2}\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$. For a test of significance $\alpha = 0.05$, we thus take $c = \sigma_0^2 \chi^2_{n,0.95}$.

4. **Suppose $X_1, \dots, X_{10}$ are independent and identically distributed $N(\mu, \sigma^2 = 20)$ random quantities, and we want to test the hypothesis**

$$H_0 : \mu = 100 \quad \text{versus} \quad H_1 : \mu = \mu_1 < 100.$$

(a) **Find the critical value for a size $\alpha = 0.05$ test.**

Since $X_1, \dots, X_{10}$ are independent identically distributed $N(\mu, 20)$ random quantities, $\overline{X} \sim N(\mu, 20/10)$. Given this pair of hypotheses, we will reject H_0 when we see unusually small values of $\overline{x}$, and so the critical value c for a size $\alpha = 0.05$ test, satisfies

$$\begin{aligned}
0.05 &= P(\text{Reject} H_0 \mid H_0 \text{ true}) \\
&= P\{\overline{X} \leq c \mid \overline{X} \sim N(100, 2)\} \\
&= P\left\{\frac{\overline{X} - 100}{\sqrt{2}} \leq \frac{c - 100}{\sqrt{2}} \;\middle|\; \overline{X} \sim N(100, 2)\right\} \\
&= \Phi\left(\frac{c - 100}{\sqrt{2}}\right) \\
\Rightarrow \quad \frac{c - 100}{\sqrt{2}} &= -1.645 \\
\Rightarrow \quad c &= 100 - 1.645\sqrt{2}
\end{aligned}$$

(b) **Calculate the corresponding probability of a Type II error as a function of μ_1, and roughly plot these probabilities against μ_1. Summarise what you deduce about the relative difficulties of testing for μ_1 close to 100 or μ_1 far from 100.**

The corresponding probability of a Type II error (as a function of μ_1) is

$$\begin{aligned}
P(\text{Type II error}) &= P(\text{ Do not reject } H_0 \mid H_0 \text{ false}) \\
&= P\{\overline{X} > c \mid \overline{X} \sim N(\mu_1, 2)\} \\
&= P\{\overline{X} > 100 - 1.645\sqrt{2} \mid \bar{X} \sim N(\mu_1, 2)\} \\
&= P\left\{\frac{\overline{X} - \mu_1}{\sqrt{2}} > \frac{100 - 1.645\sqrt{2} - \mu_1}{\sqrt{2}} \;\middle|\; \overline{X} \sim N(\mu_1, 2)\right\} \\
&= 1 - P\left\{Z < \frac{100 - 1.645\sqrt{2} - \mu_1}{\sqrt{2}} \;\middle|\; Z \sim N(0, 1)\right\} \\
&= 1 - \Phi\left(\frac{100 - \mu_1}{\sqrt{2}} - 1.645\right)
\end{aligned}$$

When μ_1 is close to 100, $(100 - \mu_1)$ is small, and the probability of a Type II error is close to $1 - \Phi(-1.645) = 0.95$. As μ_1 gets smaller, $(100 - \mu_1)$ gets larger, and hence the probability of a Type II error decreases. This should also be our intuition for the problem; when μ_1 is close to 100, it is hard to test accurately, but when the two sampling distributions are well separated, fewer errors will occur.

5. **Suppose $X_1, \ldots, X_n$ are independent and identically distributed $Exp(\lambda)$ random quantities, so $f(x|\lambda) = \lambda \exp(-\lambda x)$. We wish to test the hypothesis**

$$H_0 : \lambda = \lambda_0 \quad \text{versus} \quad H_1 : \lambda = \lambda_1$$

where $\lambda_1 > \lambda_0$.

(a) **Use the Neyman-Pearson lemma to find the test of significance α with the largest power. Try to ensure that your final test statistic is as simple as possible.**

If $X_1, \ldots, X_n$ are iid $Exp(\lambda)$ then the likelihood is given by

$$L(\lambda) \;=\; \prod_{i=1}^{n} \lambda \exp(-\lambda x_i) \;=\; \lambda^n \exp(-\lambda n \overline{x}).$$

We consider $\lambda(x_1, \ldots, x_n; \lambda_0, \lambda_1) = L(\lambda_0)/L(\lambda_1)$. We have

$$\begin{aligned}\frac{L(\lambda_0)}{L(\lambda_1)} &= \frac{\lambda_0^n \exp(-\lambda_0 n \overline{x})}{\lambda_1^n \exp(-\lambda_1 n \overline{x})} \\ &= \left(\frac{\lambda_0}{\lambda_1}\right)^n \exp\{(\lambda_1 - \lambda_0) n \overline{x}\}.\end{aligned}$$

The critical region given by the Neyman-Pearson lemma is

$$\begin{aligned}C^* &= \left\{(x_1, \ldots, x_n) : \left(\frac{\lambda_0}{\lambda_1}\right)^n \exp\{(\lambda_1 - \lambda_0) n \overline{x}\} \leq k\right\} \\ &= \{(x_1, \ldots, x_n) : \overline{x} \leq c\}\,.\end{aligned}$$

Notice that this makes intuitive sense as $E(\overline{X}|\lambda) = 1/\lambda$ and for $\lambda_1 > \lambda_0$ we have $1/\lambda_0 > 1/\lambda_1$ so small values of $\overline{x}$ indicate H_1.

(b) **If you knew the sampling distribution of your test statistic, how would you find the critical value for a test of significance $\alpha = 0.05$? You do not need to evaluate this critical value, but express it in terms of a quantile of the distribution of your test statistic.**

In order to find c, we need the sampling distribution of $\overline{X}$ under H_0 and $P(\overline{X} \leq c|H_0 \text{ true}) = \alpha$ so that, for $\alpha = 0.05$, c is the 0.05-quantile of $\overline{X}$.