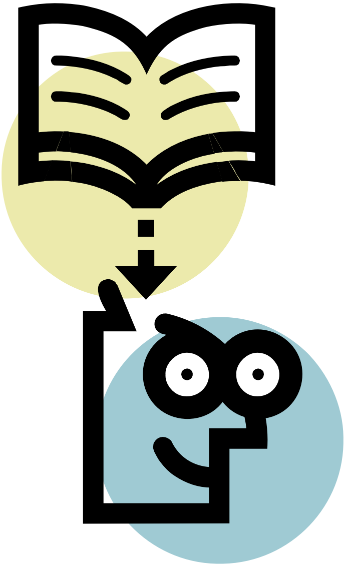
Probability Theory

Reminder

- What was the most memorable thing we learned last time?



Independence

- A and B are independent if:
  - $P(A \cap B) = P(A)\,P(B)$
  - Or, $P(A|B) = P(A)$
- For continuous variables:
  - $f(x|y) = f(x)$ for all y

Conditional Independence

- A and B are conditionally independent (given C) if:
  - $P(A \cap B|C) = P(A|C)\,P(B|C)$
  - Or equivalently, $P(A|B \cap C) = P(A|C)$
- In this case:
  - B has no new information about A
  - After we know C
- Influence diagram for this case?

Conditional Independence

- Does conditional independence imply dependence? NO!
- Does independence imply conditional independence? NO!

Conditional Independence

- Conditional independence of X and Y given Z=z (even for all z) does *not* imply independence of X and Y
- Independence of X and Y does *not* imply conditional independence given Z

Example

State of Economy	Good	Bad
Probability	.6	.4
Success of Business	.9	.1
Election of Republican	.7	.3

- Business success and Republican win are *conditionally* independent:
  - But not independent!

Example (continued)

- P(Republican | success) =

  $P(\text{Republican} \cap \text{success})/P(\text{success}) =$

  $[P(\text{Rep.} \cap \text{succ.}|G)P(G) + P(\text{Rep.} \cap \text{succ.}|B)P(B)]/$

  $[P(\text{success}|G)P(G) + P(\text{success}|B)P(B)] \approx .67$

- P(Republican) =

  $P(\text{Rep.}|G)P(G) + P(\text{Rep.}|B)P(B) = .54$

- Success does not influence the election:
  - But provides evidence about the economy!

Conditional Independence

- What about the converse?
- Independence of X and Y does *not* imply conditional independence given Z

Example

State of Economy	Peace in Middle East	Election of Republican
Good (.6)	Good (.8)	.9
	Bad (.2)	.5
Bad (.4)	Good (.8)	.5
	Bad (.2)	.1

Example (continued)

- Good economy and peace in the middle east are *independent*:
  - But not conditionally independent, given election of a Republican

Example (continued)

- P(Good economy | middle east bad $\cap$ Republican) =

  P(Good GNP $\cap$ no peace $\cap$ Republican)/ P(middle east bad $\cap$ Republican)

  = .6 (.2) (.5)/[.6 (.2) (.5) +.4 (.2) (.1)] $\approx$ .88

- P(Good economy | Republican) = .71 =

  P(Good GNP $\cap$ Republican)/P(Republican)

  = [.6 (.8) (.9) + .6 (.2) (.5)]/

  [.6 (.8) (.9) + .6 (.2) (.5) + .4 (.8) (.5) + .4 (.2) (.1)]

Basic Concepts

- Random variable:
  - Takes on a value for each point in the sample space

Probability Mass Function

- $P_X(x_i) = P(X = x_i)$
  - $\sum P_X(x_i) = 1$
  - $0 \leq P_X(x_i) \leq 1$

Probability Density Function

- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
  - $f_X(x) \geq 0$
  - $P(y \leq X \leq z) = \int_y^z f_X(x)dx$
  - $P(x \leq X \leq x+dx) \approx f_X(x)\, dx$

Probability Density Function

- $F_X(y) = \int_{-\infty}^{y} f_X(x)dx$
- Monotonic:
  - $z \geq y \Rightarrow F_X(z) \geq F_X(y)$
- $P(y \leq X \leq z) = F_X(z) - F_X(y)$
- $\frac{d}{dx}F_X(y) = f_X(y)$

Expectation

- $E(X) = \int x \, f_X(x) \, dx$
- $E[g(X)] = \int g(x) \, f(x) \, dx$
  - What happens when g is linear?
- Moments:
  - $E(X^n)$
- Central moments (about the mean):
  - $E[(X-E(X))^n]$

Measures of Spread

- Variance:
  - $Var(X) = E[(X-E(X))^2] = E(X^2) - [E(X)]^2$
    - $Var(kX) = k^2 \, Var(X)$
- Standard deviation:
  - $\sigma(X) = \sqrt{Var(X)}$, $\sigma(kX) = k \, \sigma(X)$
- Coefficient of variation:
  - $CV(X) = \sigma(X)/E(X)$, $CV(kX) = CV(X)$
    - *Unit-less* measure of spread!

Joint Probability Distributions

- $f_{X,Y}(x,y)$
- Marginal distributions:
  - $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$
  - $F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$

Conditional Distributions

- $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y)$
- Conditional expectations:
  - $E(X|Y) = \int x \, f_X(x|y) \, dx$
  - $E(X) = \int E(X|Y=y) \, f_Y(y) \, dy$
- Note:
  - $Var(X) \neq \int Var(X|Y=y) \, f_Y(y) \, dy$
  - ***Why not?***

Independence

- $f_{X|Y}(x|y) = f_X(x)$ for all x,y
  - Equivalent, $F_{X,Y}(x,y) = F_X(x)\, F_Y(y)$ for all x,y
- Implications:
  - $E(XY) = E(X)\, E(Y)$
  - $E[g(X)\, h(Y)] = E[g(X)]\, E[h(Y)]$ for all g,h

Linear Independence

- $E(XY) = E(X)\,E(Y)$
  - Independence implies linear independence, but *not* vice versa!
- Example:
  - If $X=1$, $Y = +/-10$ with probability 0.5
  - If $X=2$, $Y = +/-1$ with probability 0.5
    - $E(XY) = E(X)\,E(Y) = 0$
    - But X and Y are not independent!

Covariance and Correlation

- Covariance:
  - $Cov(X<Y) = E(XY) - E(X)\, E(Y)$
- Correlation coefficient:
  - $\rho = Cov(X,Y)/(\sigma_X\, \sigma_Y)$
  - $-1 \leq \rho \leq 1$