MA20033 - Question Sheet Seven

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Attempt questions 1-3. Hand in by 5.00pm Friday 3 December 2004 either to me, or in the envelope on my door, 1W4.8. Questions 4 and 5 are for tutorial discussion, but I'm happy to also mark these if you wish to attempt them.

1. Suppose $X_1, \ldots, X_{10}$ are independent and identically distributed $N(\mu, \sigma^2 = 25)$ random quantities.

   (a) Carry out the following one-sided test by stating a test statistic and finding the critical value of the test:

   $$H_0 : \mu = 105 \quad \text{versus} \quad H_1 : \mu > 105$$

   (b) Carry out this test at the 5% significance level when a sample mean $\overline{x} = 108$ is obtained.

   (c) Evaluate the power of this test at the points $\mu = 106$ and $\mu = 110$.

2. Again suppose $X_1, \ldots, X_{10}$ are independent and identically distributed $N(\mu, \sigma^2 = 25)$ random quantities.

   (a) Carry out the following two-sided test by stating a test statistic and finding the critical value of the test:

   $$H_0 : \mu = 105 \quad \text{versus} \quad H_1 : \mu \neq 105$$

   (b) Carry out this test at the 5% significance level when a sample mean $\overline{x} = 108$ is obtained.

   (c) Evaluate the power of this test at the points $\mu = 106$ and $\mu = 110$.

   (d) Would you have expected the power to be greater or smaller than that of the one-sided test?

3. Suppose $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, 1)$ random quantities, and you want a test with significance level 5% of the one-sided hypotheses

   $$H_0 : \mu = 0 \quad \text{versus} \quad H_1 : \mu < 0$$

   (a) Find the critical value k such that the null hypothesis will be rejected if $\overline{x} < k$ (this critical value will depend on the sample size through the variance of $\overline{X}$).

(b) Suppose that you require your test to have power of at least 0.95 when μ is actually equal to -0.25. What is the probability that H_0 is not rejected when $\mu = -0.25$ (as a function of n)? So how big must the sample size n be to meet this power requirement?

4. Suppose $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, \sigma^2)$ random quantities.

   (a) What is the sampling distribution of S^2?

   (b) Set up a hypothesis test to determine whether the variability of some process has changed: Test the hypothesis $H_0 : \sigma^2 = 10$ against $H_1 : \sigma^2 \neq 10$, when a sample of size 20 yields an observed value of $s^2 = 13.8$.

5. The Neyman-Pearson lemma states that of all tests with significance α of the simple hypotheses

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

the test which uses the critical region

$$C^* \quad = \quad \{(x_1, \ldots, x_n) : \lambda(x_1, \ldots, x_n; \theta_0, \theta_1) \leq k\},$$

where $\lambda(x_1, \ldots, x_n; \theta_0, \theta_1) = f(x_1, \ldots, x_n | \theta = \theta_0)/f(x_1, \ldots, x_n | \theta = \theta_1)$ and k is a constant chosen to ensure the significance is α, is the one with the largest power. Denote $(X_1, \ldots, X_n)$ by $\underline{X}$ and $(x_1, \ldots, x_n)$ by $\underline{x}$.

   (a) Show that for any set $A \subset C^*$,

$$P(\underline{X} \in A | \theta = \theta_0) \ \leq \ kP(\underline{X} \in A | \theta = \theta_1).$$

   (b) Show that for any set $A \subset \overline{C^*}$ (the compliment of C^*)

$$P(\underline{X} \in A | \theta = \theta_0) \ > \ kP(\underline{X} \in A | \theta = \theta_1).$$

   (c) Let C be some other critical region with significance level α. Show that

$$P(\underline{X} \in C^* | \theta) - P(\underline{X} \in C | \theta) \quad = \quad P(\underline{X} \in (C^* \cap \overline{C}) | \theta) - P(\underline{X} \in (C \cap \overline{C^*}) | \theta),$$

   where $\overline{C}$ denotes the compliment of C.

   (d) Hence show that

$$P(\underline{X} \in C^* | \theta = \theta_1) - P(\underline{X} \in C | \theta = \theta_1) \ \geq \ 0.$$

   You have now proved the Neyman-Pearson lemma.