Chapter 3:
Basic Probability Concepts

- **Probability:**
- is a measure (or number) used to measure the chance of the occurrence of some event. This number is between 0 and 1.
- **An experiment:**
- is some procedure (or process) that we do.
- **Sample Space:**
- The set of all possible outcomes of an experiment is called the sample space (or Universal set) and is denoted by Ω

An Event:

is a subset of the sample space Ω.

- $\phi \subseteq \Omega$ is an event (impossible event)
- $\Omega \subseteq \Omega$ is an event (sure event)

Example:

Experiment: Selecting a ball from a box containing 6 balls numbered 1,2,3,4,5 and 6.

This experiment has 6 possible outcomes

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- Consider the following events:

E_1: getting an event number $= \{2, 4, 6\} \subseteq \Omega$

E_2: getting a number less than 4 $= \{1, 2, 3\} \subseteq \Omega$

E_3 =getting 1 or 3 $= \{1, 3\} \subseteq \Omega$

E_4 =getting an odd number $= \{1, 3, 5\} \subseteq \Omega$

Notation $n(\Omega) =$ no. of outcomes (elements) in Ω

$n(E_i) =$ no. of outcomes (elements) in E_i

Equally likely outcomes:

The outcomes of an experiment are equally likely if the occurrences of the outcomes have the same chance.

Probability of an event:

- **If the experiment has *N* equally likely outcomes, then the probability of the event *E* is:**

$$P(E)=\frac{n(E)}{n(\Omega)}=\frac{n(E)}{N}=\frac{\text{no. of outcomes in } E}{\text{no. of outcomes in } \Omega}$$

Example: In the ball experiment in the previous example, suppose the ball is selected randomly.

$$\Omega = \{1, 2, 3, 4, 5, 6\} \quad ; \quad n(\Omega) = 6$$
$$E_1 = \{2, 4, 6\} \quad ; \quad n(E_1) = 3$$
$$E_2 = \{1, 2, 3\} \quad ; \quad n(E_2) = 3$$
$$E_3 = \{1, 3\} \quad ; \quad n(E_3) = 2$$

The outcomes are equally likely.

$$\therefore P(E_1)=\frac{3}{6}, \quad P(E_2)=\frac{3}{6}, \quad P(E_3)=\frac{2}{6},$$

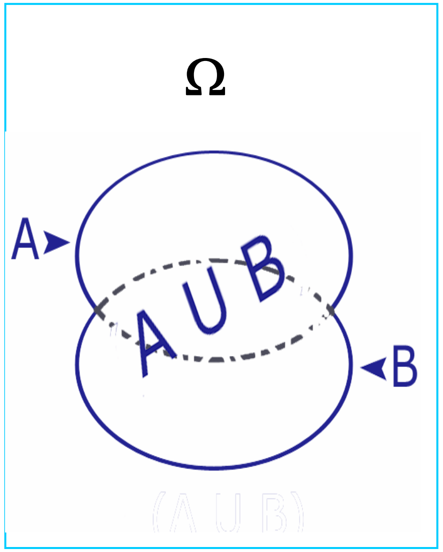
Some Operations on Events:

Let A and B be two events defined on Ω

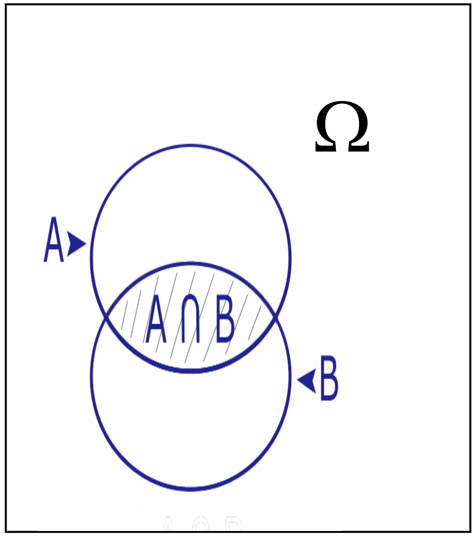
Union: $A \cup B$

$A \cup B$ **Consists of all outcomes in *A* or in *B* or in both *A* and *B*.**

$A \cup B$ **Occurs if A occurs, or B occurs, or both A and B occur.**



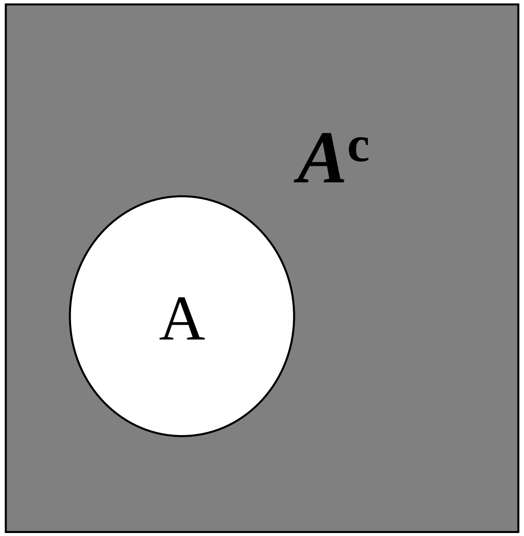
Intersection: $A \cap B$



$A \cap \cdot B$ Consists of all outcomes in both *A* **and** *B*.

$A \cap B$ Occurs if both *A* **and** *B* occur .

Complement: $\boldsymbol{A^c}$



- A^c is the complement of *A*.
- A^c consists of all outcomes of Ω but are not in *A*.
- A^c occurs if *A* does not.

Example:

Experiment: Selecting a ball from a box containing 6 balls numbered 1, 2, 3, 4, 5, and 6 randomly.

Define the following events:

$E_1 = \{2, 4, 6\}$ = getting an even number.

$E_2 = \{1, 2, 3\}$ = getting a number < 4.

$E_3 = \{1, 3\}$ = getting 1 or 3.

$E_4 = \{1, 3, 5\}$ = getting an odd number.

(1) $E_1 \cup E_2 = \{1, 2, 3, 4, 6\}$

= getting an even no. **or** a no. less than 4.

$$P(E_1 \cup E_2) = \frac{n(E_1 \cup E_2)}{n(\Omega)} = \frac{5}{6}$$

(2) $E_1 \cup E_4 = \{ 1, 2, 3, 4, 5, 6 \} = \Omega$

= getting an even no. **or** an odd no.

$$P(E_1 \cup E_4) = \frac{n(E_1 \cup E_4)}{n(\Omega)} = \frac{6}{6} = 1$$

Note: $E_1 \cup E_4 = \Omega$

E_1 and E_4 are called exhaustive events.

(3) $E_1 \cap E_2 = \{ 2 \}$

= getting an even no. **and** a no.less than 4.

$$P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(\Omega)} = \frac{1}{6}$$

(4) $E_1 \cap E_4 = \phi$

= getting an even no. **and** an odd no.

$$P(E_1 \cap E_4) = \frac{n(E_1 \cap E_4)}{n(\Omega)} = \frac{n(\phi)}{6} = \frac{0}{6} = 0$$

Note: $E_1 \cap E_4 = \phi$

E_1 and E_4 are called disjoint (or mutually exclusive) events.

(5) E_1^c = not getting an even no. = {1, 3, 5}

= getting an odd no.

= E_4

Notes:

1. The event A_1 , A_2 , …, A_n are exhaustive events if

$$A_1 \cup A_2 \cup \ldots \cup A_n = \Omega$$

2. The events A and B are disjoint (or mutually exclusive) if

$$A \cap B = \phi$$

In this case :

(i) $P(A \cap B) = 0$

(ii) $P(A \cup B) = P(A) + P(B)$

3. $A \cup A^c = \Omega$, A and A^c are exhaustive events.
$A \cap A^c = \phi$, A and A^c are disjoint events.

4. $n(A^c) = n(\Omega) - n(A)$

$P(A^c) = 1 - P(A)$

General Probability Rules:-

1. $0 \le P(A) \le 1$

2. $P(\Omega) = 1$

3. $P(\phi) = 0$

4. $P(A^c) = 1 - P(A)$

5. For any events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

6. For disjoint events A and B

$$P(A \cup B) = P(A) + P(B)$$

7. For disjoint events $E_1, E_2, \ldots, E_n$

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

2.3. Probability applied to health data:-

Example 3.1:

630 patients are classified as follows : (Simple frequency table)

Blood Type	O (E_1)	A (E_2)	B (E_3)	AB (E_4)	Total
No. of patients	284	258	63	25	630

- Experiment: Selecting a patient at random and observe his/her blood type.
- This experiment has 630 equally likely outcomes

$$\therefore n(\Omega) = 630$$

Define the events :

E_1 = The blood type of the selected patient is O
E_2 = The blood type of the selected patient is A
E_3 = The blood type of the selected patient is B
E_4 = The blood type of the selected patient is AB

$n(E_1) = 284$, $n(E_2) = 258$, $n(E_3) = 63$, $n(E_4) = 25$.

$$P(E_1) = \frac{284}{630} \quad , \quad P(E_2) = \frac{258}{630} \quad , \quad P(E_3) = \frac{63}{630} \quad , \quad P(E_4) = \frac{25}{630}$$

$E_2 \cup E_4 =$ the blood type of the selected patients is A **or** AB

$$P(E_2 \cup E_4) = \begin{cases} \dfrac{n(E_2 \cup E_4)}{n(\Omega)} = \dfrac{258+25}{630} = \dfrac{283}{630} = 0.4492 \\ or \\ P(E_2) + P(E_4) = \dfrac{258}{630} + \dfrac{25}{630} = \dfrac{283}{630} = 0.4492 \end{cases}$$

(since $E_2 \cap E_4 = \phi$)

Notes:

1. E_1 ,E_2 ,E_3 ,E_4 are mutually disjoint, $E_i \cap E_j = \phi \quad (i \neq j)$

2. E_1 ,E_2 ,E_3 ,E_4 are exhaustive events, .$E_1 \cup E_2 \cup E_3 \cup E_4 = \Omega$

Example 3.2:

Smoking Habit

Age	Daily (B_1)	Occasionally (B_2)	Not at all (B_3)	Total
20 - 29 (A_1)	31	9	7	47
30 - 39 (A_2)	110	30	49	189
40 - 49 (A_3)	29	21	29	79
50+ (A_4)	6	0	18	24
Total	176	60	103	339

Experiment: Selecting a physician at random

$n(\Omega) = 339$ equally likely outcomes

Events:

- A_3 = the selected physician is aged 40 - 49

$$P(A_3) = \frac{n(A_3)}{n(\Omega)} = \frac{79}{339} = 0.2330$$

- B_2 = the selected physician smokes occasionally

$$P(B_2) = \frac{n(B_2)}{n(\Omega)} = \frac{60}{339} = 0.1770$$

- $A_3 \cap B_2 =$ the selected physician is aged 40-49 **and** smokes occasionally.

$$P(A_3 \cap B_2) = \frac{n(A_3 \cap B_2)}{n(\Omega)} = \frac{21}{339} = 0.06195$$

$A_3 \cup B_2 =$ the selected physician is aged 40-49 **or** smokes occasionally (**or** both)

$$P(A_3 \cup B_2) = P(A_3) + P(B_2) - P(A_3 \cap B_2)$$

$$= \frac{79}{339} + \frac{60}{339} - \frac{21}{339}$$

$$= 0.233 + 0.177 - 0.06195 = = 0.3481$$

$\cdot A_4^c =$ the selected physician is **not** 50 years or older.

$$P(A_4^c) = 1 - P(A_4)$$

$$= 1 - \frac{n(A_4)}{n(\Omega)} = 1 - \frac{24}{339} = 0.9292$$

$A_2 \cup A_3 =$ the selected physician is aged 30-49 **or** is aged 40-49

= the selected physician is aged 30-49

$$\begin{cases} P(A_2 \cup A_3) = \dfrac{n(A_2 \cup A_3)}{n(\Omega)} = \dfrac{189+79}{339} = \dfrac{286}{339} = 0.7906 \\ or \\ P(A_2 \cup A_3) = P(A_2) + P(A_3) = \dfrac{189}{339} + \dfrac{79}{339} = 0.7906 \end{cases}$$

(Since $A_2 \cap A_3 = \phi$)

3.3. (Percentage/100) as probabilities and the use of venn diagrams:

$$P(E) = \frac{n(E)}{n(\Omega)} =$$

$n(\Omega) =$ **??** **unknown** $n(E) =$ **??** **unknown**

, , $\%(E) =$ Percentage of elements of ***E*** relative

to the elements of Ω, $\boldsymbol{n(\Omega)}$, is known.

$$\%(E) = \frac{n(E)}{n(\Omega)} \times 100\%$$

Example 3.3: (p.72)

A population of pregnant women with:

- 10% of the pregnant women delivered prematurely.
- 25% of the pregnant women used some sort of medication.
- 5% of the pregnant women delivered prematurely <u>and</u> used some sort of medication.

Experiment : Selecting a woman randomly from this population.

Define the events:

- D = The selected woman delivered prematurely.
- M = The selected women used some sort of medication.

$D \cap M =$ The selected woman delivered prematurely <u>and</u> used some sort of medication.

$$\%(D) = 10\% \qquad \%(M) = 25\% \qquad \%(D \cap M) = 5\%$$

$$\therefore P(D) = \frac{\%(D)}{100\%} = \frac{10\%}{100\%} = 0.1$$

$$P(M) = \frac{\%(M)}{100\%} = \frac{25\%}{100\%} = 0.25$$

$$P(D \cap M) = \frac{\%(D \cap M)}{100\%} = \frac{5\%}{100\%} = 0.05$$

A Venn diagram:

$$P(D) = 0.1$$

$$P(M) = 0.25$$

$$P(D \cap M) = 0.05$$

$$P(D^c \cap M) = 0.2$$

$$P(D \cap M^c) = 0.05$$

$$P(D^c \cap M^c) = 0.70$$

$$P(D \cup M) = 0.30$$

Probability given by a Venn diagram

A Two-way table:

	$\boldsymbol{M}$	M^c	Total
D	**0.05**	0.05	**0.10**
D^c	0.20	0.70	0.90
Total	**0.25**	0.75	1.00

Probabilities given by a two-way table.

Calculating probabilities of some events:

M^c = The selected woman did not use medication

$$P(M^c) = 1 - P(M) = 1 - 0.25 = 0.75$$

$D^c \cap M^c =$ the selected woman din not deliver prematurely <u>and</u> did not use medication.

$$\boldsymbol{P\left(D^c \cap M^c\right) = 1 - P\left(D \cup M\right) = ??}$$

$D \cup M =$ the selected woman delivered prematurely <u>or</u> used some medication.

$$P\left(D \cup M\right) = P\left(D\right) + \left(M\right) - P\left(D \cap M\right)$$

$$= 0.1 + 0.25 - 0.05 = 0.3$$

$$\because P\left(D^c \cap M^c\right) = 1 - P\left(D \cup M\right) = 1 - 0.3 = 0.7$$

<u>Note:</u>

From the Venn diagram, it is clear that:

$$P\left(D\right) = P\left(D \cap M\right) + P\left(D \cap M^c\right)$$

$$P\left(M\right) = P\left(D \cap M\right) + P\left(D^c \cap M\right)$$

$$P\left(D \cap M^{c}\right) = P(D) - P(D \cap M)$$

$$P\left(D^{c} \cap M\right) = P(M) - P(D \cap M)$$

$$P\left(D^{c} \cap M^{c}\right) = 1 - P(D \cup M)$$

3.4. Conditional Probability:

- The conditional probability of the event A given the event B is defined by:

$$P(A|B)=\frac{P(A\cap B)}{P(B)} \qquad ; \; P(B)\neq 0$$

- $P(A \mid B)$ = the probability of the event A if we know that the event B has occurred.

Note:

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

$$=\frac{n(A\cap B)/n(\Omega)}{n(B)/n(\Omega)}$$

$$\therefore P(A|B)=\frac{n(A\cap B)}{n(B)}$$

$$P(A \cap B) = P(B)P(A \mid B)$$
$$P(A \cap B) = P(A)P(B \mid A)$$

multiplication rules

Example:

Smoking Habbit

Age

	Daily (B_1)	Occasionally (B_2)	Not at all (B_3)	Total
20-29 A_1)	31	9	7	47
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50+ (A_4)	6	0	18	24
	176	60	103	339

For calculating $P(A \mid B)$, we can use

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad or \quad P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Using the restricted table directly

$$P(B_1) = \frac{176}{339} = 0.519$$

$$P(B_1 \mid A_2) = \frac{P(B_1 \cap A_2)}{P(A_2)} = \frac{0.324484}{0.557522} = 0.5820$$

$$\boldsymbol{P(B_1 \cap A_2) = \frac{n(B_1 \cap A_2)}{n(\Omega)}} = \frac{110}{339} = 0.324484$$

$$P(A_2) = \frac{n(A_2)}{n(\Omega)} = \frac{189}{339} = 0.557522$$

OR

$$P(B_1 \mid A_2) = \frac{n(B_1 \cap A_2)}{n(A_2)}$$

$$= \frac{110}{189} = 0.5820$$

Notice that $P(B_1) < P(B_1 \mid A_2)$ **!! … What does this mean?**

Independent Events

There are 3 cases

(1) $\boldsymbol{P(A \mid B) > P(A)}$

which means that **knowing *B* increases the probability of occurrence of *A* .**

(2) $$P(A \mid B) < P(A)$$

which means that **knowing *B* decreases the probability of occurrence of *A* .**

(3) $$P(A \mid B) = P(A)$$

which means that **knowing *B* has no effect on the probability of occurrence of *A* .**

In this case A is independent of B.

Independent Events:

Two events A and B are independent if one of the following conditions is satisfied:

$$(i) \quad P(A \mid B) = P(A)$$

$$\Leftrightarrow (ii) \quad P(B \mid A) = P(B)$$

$$\Leftrightarrow (iii) \quad P(B \cap A) = P(A)P(B)$$

⇧

(multiplication rule)

Example:

In the previous , $\boldsymbol{A}_2$ and $\boldsymbol{B_1}$ are **not independent** because:

$$P(B_1) = 0.5192 \neq P(B_1 \mid A_2) = 0.5820$$

also

$$P(B_1 \cap A_2) = 0.32448 \neq P(B_1)P(A_2) = 0.28945$$

Combinations:

- **Notation:** n factorial is denoted by $n!$ and is defined by:

$$n! = n(n-1)(n-2)\cdots(2)(1) \qquad \textit{for} \quad n \geq 1$$

$$0! = 1$$

Example: 5! = (5) (4) (3) (2) (1) = 120

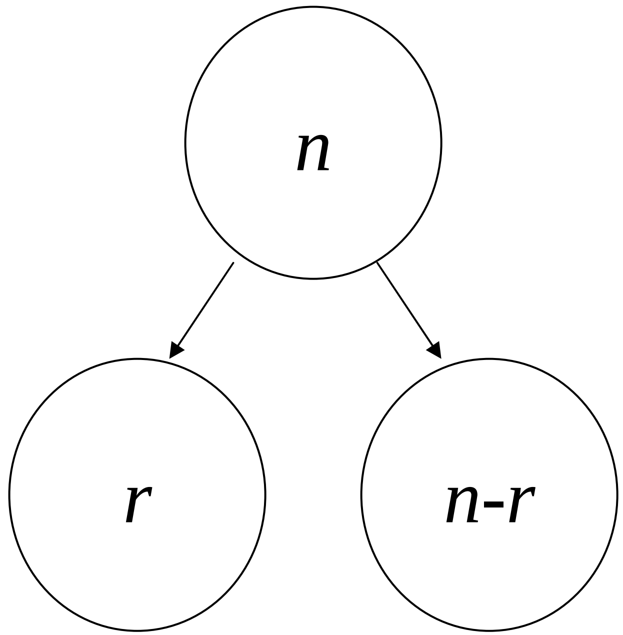
- **Combinations:**

The number of different ways for selecting r objects from n distinct objects is denoted by $\binom{n}{r}$ and is given by:

$$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}; \qquad r = 0, 1, 2, \ldots, n$$

$\binom{n}{r}$ is read as " n " choose " r ".

$$\binom{n}{n}=1 \qquad \binom{n}{0}=1 \qquad \binom{n}{r}=\binom{n}{n-r}$$



Example 3.9:

If we have 10 equal–priority operations and only 4 operating rooms, in how many ways can we choose the 4 patients to be operated on first?

Answer:

The number of different ways for selecting 4 patients from 10 patients is

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10!}{4!\,6!} = \frac{(10)(9)(8)\cdots(2)(1)}{(4)(3)(2)(1)\quad(6)(5)(4)(3)(2)(1)}$$

$$= 210 \qquad\qquad (\textit{different ways})$$