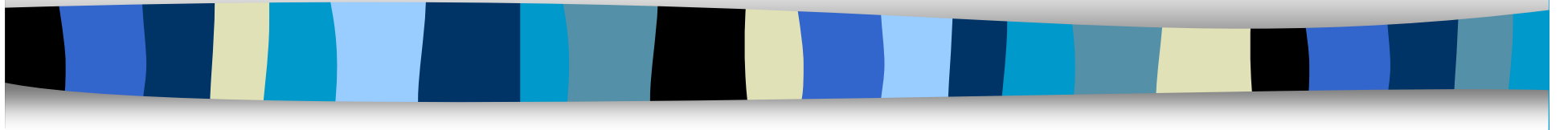


Chapter 4: Probability Distributions





Chapter 4: Probability Distributions

Some events can be defined using random variables.

Random variables $\left\{ \begin{array}{l} \textit{Discrete Random Variables} \\ \textit{Continuous Random Variables} \end{array} \right.$

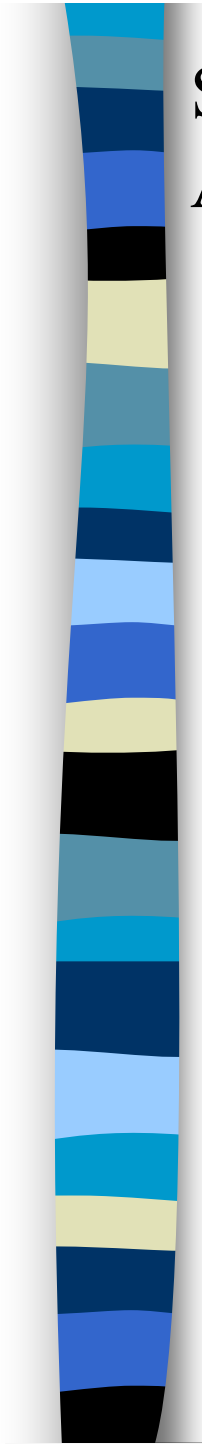
4.2. Probability Distributions of Discrete R.V.'s:-

Examples of discrete r v.'s

- The no. of patients visiting KKUH in a week.
- The no. of times a person had a cold in last year.

Example: consider the following discrete random variable.

X = The number of times a person had a cold in January 1998 in Saudi Arabia.



Suppose we are able to count the no. of people in Saudi Arabia for which $X = x$

x	Frequency of x
(no. of times a person had a cold in January 1998 in S. A.)	(no. of people who had a cold in January 1998 in S.A.)
0	10,000,000
1	3,000,000
2	2,000,000
3	1,000,000
Total	$N = 16,000,000$

Simple frequency table of no. of times a person had a cold in January 1998 in Saudi Arabia.



Experiment: Selecting a person at random

Define the event:

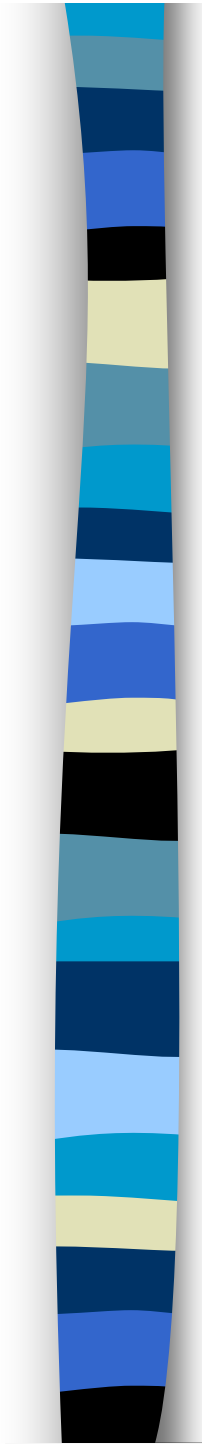
$(X = x) =$ The event that the selected person had a cold in January 1998 x times.

In particular,

$(X = 2) =$ The event that the selected person had a cold in January 1998 two times.

For this experiment, there are $n(\Omega) = 16,000,000$ equally likely outcomes.

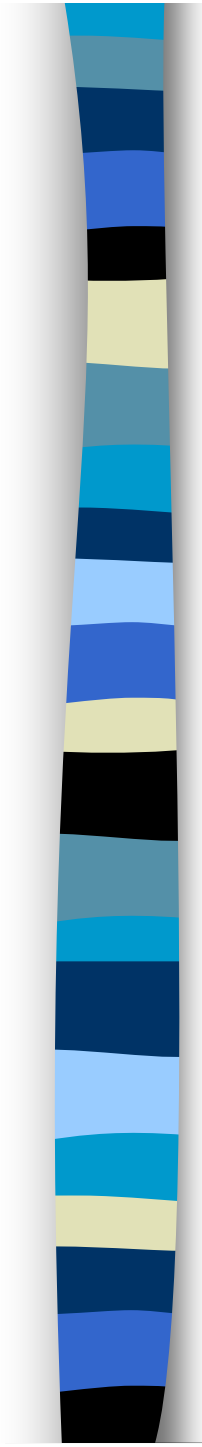
$$\therefore P(X = x) = \frac{n(X = x)}{n(\Omega)} = \frac{\left(\begin{array}{l} \text{no. of people who had } x \\ \text{colds in January 1998} \end{array} \right)}{16,000,000}$$



x	freq. of x $n(X=x)$	$P(X = x)$ $= n(X = x)/1600000$
0	10000000	0.6250
1	3000000	0.1875
2	2000000	0.1250
3	1000000	0.0625
Total	16000000	1.0000

Note:

$$\begin{aligned} P(X = x) &= \frac{n(X = x)}{16000000} \\ &= \text{Relative Frequency} \\ &= \frac{\text{frequency}}{16000000} \end{aligned}$$

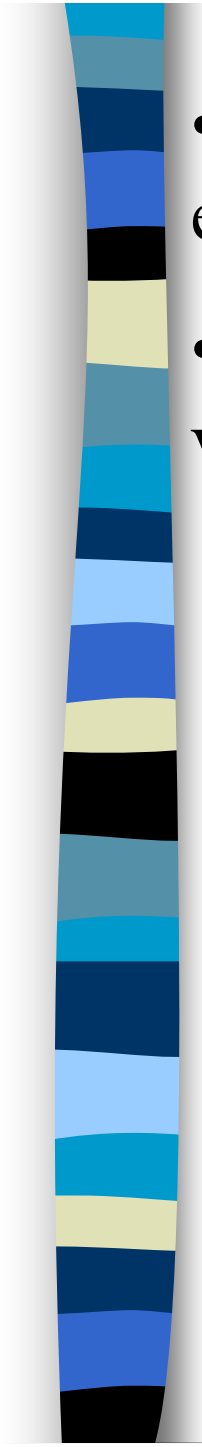


x	$P(X = x)$
0	0.6250
1	0.1874
2	0.1250
3	0.0625
Total	1.0000

This table is called the probability distribution of the discrete random variable X .

Notes:

- $(X=0)$, $(X=1)$, $(X=2)$, $(X=3)$ are mutually exclusive (disjoint) events.



- $(X=0)$, $(X=1)$, $(X=2)$, $(X=3)$ are mutually exhaustive events

- The probability distribution of any discrete random variable x must satisfy the following two properties:

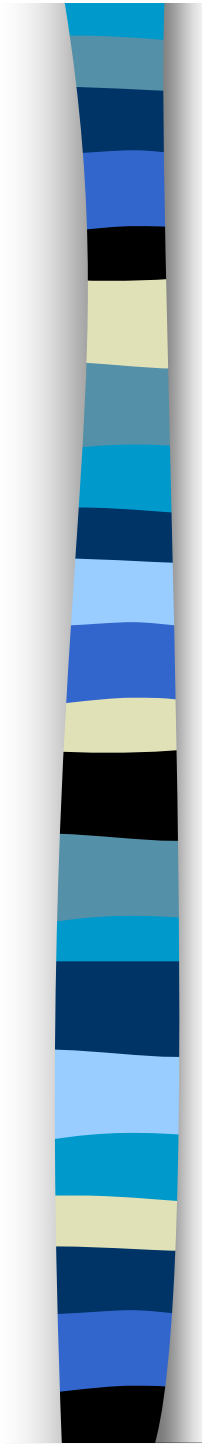
$$(1) \quad 0 \leq P(X = x) \leq 1$$

$$(2) \quad \sum_x P(X = x) = 1$$

- Using the probability distribution of a discrete r.v. we can find the probability of any event expressed in term of the r.v. X .

Example:

Consider the discrete r.v. X in the previous example.



x	$P(X=x)$
0	0.6250
1	0.1875
2	0.1250
3	0.0625

$$(1) \quad P(X \geq 2) = P(X = 2) + P(X = 3) = 0.1250 + 0.0625 = 0.1875$$

$$(2) \quad P(X > 2) = P(X = 3) = 0.0625$$

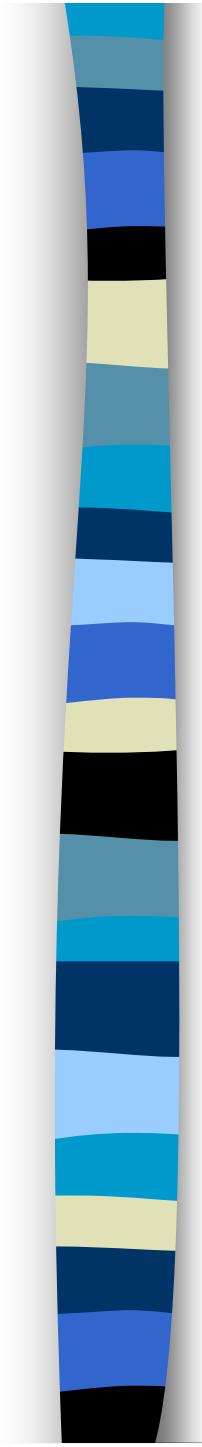
$$(3) \quad P(1 \leq X < 3) = P(X = 1) + P(X = 2) = 0.1875 + 0.1250 = 0.3125$$

$$(4) \quad P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) \\ = 0.6250 + 0.1875 + 0.1250 = 0.9375$$

Or

$$P(X \leq 2) = 1 - P((X \leq 2)^c) \\ = 1 - P(X > 2)$$

$$(5) \quad P(-1 \leq X < 2) = P(X = 0) + P(X = 1) \\ = 0.6250 + 0.1875 = 0.8125$$



(6) $P(-1.5 \leq X < 1.3) = P(X = 0) + P(X = 1)$
 $= 0.6250 + 0.1875 = 0.8125$

(7) $P(X = 3.5) = P(\phi) = 0$

(8) The probability that the selected person had at least
2 colds

$$= P(X \geq 2)$$
$$= P(X = 2) + P(X = 3) = 0.1875$$

(9) The probability that the selected person had at most
2 colds

$$= P(X \leq 2)$$
$$= 0.9375$$

(10) The probability that the selected person had more
than 2 colds

$$= P(X > 2)$$
$$= P(X = 3) = 0.0625$$



(11) The probability that the selected person had less than 2 colds

$$= P(X < 2)$$

$$= P(X = 0) + P(X = 1) = 0.8125$$

Graphical Presentation:

The probability distribution of a discrete r. v. X can be graphically presented as follows

x	$P(X=x)$
0	0.6250
1	0.1875
2	0.1250
3	0.0625

Population Mean of a Discrete Random

The mean of a discrete random variable X is denoted by μ and defined by:

$$\mu = \sum_x x P(X = x) \quad [\text{mean} = \text{expected value}]$$

Example: We wish to calculate the mean μ of the discrete r. v. X in the previous example.

x	$P(X=x)$	$xP(X=x)$
0	0.6250	0.0
1	0.1875	0.1875
2	0.1250	0.2500
3	0.0625	0.1875
Tota 1	$\mu = \sum xP(X = x)$ =0.625	$\sum P(X = x) = 1.00$

$$\mu = \sum_x xP(X = x) = (0)(0.625) + (1)(0.1875) + (2)(0.125) + (3)(0.0625) = 0.625$$

Cumulative Distributions:

The cumulative distribution of a discrete r. v. x is defined by

$$P(X \leq x) = \sum_{a \leq x} P(X = a)$$

Example: The cumulative distribution of X is the previous example is:

x	$P(X \leq x)$	-
0	0.6250	$P(X \leq 0) = P(X = 0)$
1	0.8125	$P(X \leq 1) = P(X = 0) + P(X = 1)$
2	0.9375	$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$
3	1.0000	$P(X \leq 3) = P(X = 0) + \dots + P(X = 3)$



Binomial Distribution:

- It is discrete distribution.
- It is used to model an experiment for which:
 1. The experiment has trials.
 2. Two possible outcomes for each trial :
 S = success and F = failure
 3. (boy or girl, Saudi or non-Saudi,...)
 4. The probability of success: is constant for each trial.
 5. The trials are independent; that is the outcome of one trial has no effect on the outcome of any other trial

The discrete r. v.

X = The number of successes in the n trials has a binomial distribution with parameter n and π , and we write

$$X \sim \text{Binomial}(n, \pi)$$



The probability distribution of X is given by:

$$P(X = x) = \begin{cases} \binom{n}{x} \pi^x (1 - \pi)^{n-x} & \text{for } x = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

where

$$\binom{n}{x} = \frac{n!}{x! (n-x)!}$$



We can write the probability distribution of X is a table as follows.

X	$P(X = x)$
0	$\binom{n}{0} \pi^0 (1 - \pi)^{n-0} = (1 - \pi)^n$
1	$\binom{n}{1} \pi^1 (1 - \pi)^{n-1}$
2	$\binom{n}{2} \pi^2 (1 - \pi)^{n-2}$
\vdots	\vdots
$n - 1$	$\binom{n}{n-1} \pi^{n-1} (1 - \pi)^2$
n	$\binom{n}{n} \pi^n (1 - \pi)^0 = \pi^n$
Total	1



Result:

If $X \sim \text{Binomial}(n, \pi)$, then

- The mean: $\mu = n\pi$ (expected value)
- The variance: $\sigma^2 = n\pi(1 - \pi)$

Example: 4.2 (p.106)

Suppose that the probability that a Saudi man has high blood pressure is 0.15. If we randomly select 6 Saudi men, find the probability distribution of the number of men out of 6 with high blood pressure. Also, find the expected number of men with high blood pressure.



Solution:

X = The number of men with high blood pressure in 6 men.

S = Success: The man has high blood pressure

F = failure: The man does not have high blood pressure.

• Probability of success $P(S) = \pi = 0.15$

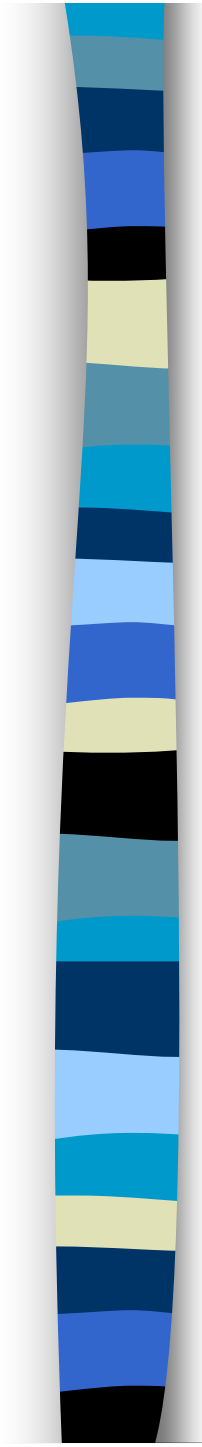
• no. of trials $n=6$

$X \sim \text{Binomial}(6, 0.15)$

$$\begin{bmatrix} \pi = 0.15 \\ 1 - \pi = 0.85 \\ n = 6 \end{bmatrix}$$

The probability distribution of X is:

$$P(X = x) = \begin{cases} \binom{6}{x} (0.15)^x (0.85)^{6-x} & ; x = 0, 1, 2, 3, 4, 5, 6 \\ 0 & ; \textit{otherwise} \end{cases}$$


$$P(X = 0) = \binom{6}{0} (0.15)^0 (0.85)^6 = (1)(0.15)^0 (0.85)^6 = 0.37715$$

$$P(X = 1) = \binom{6}{1} (0.15)^1 (0.85)^5 = (6)(0.15)(0.85)^5 = 0.39933$$

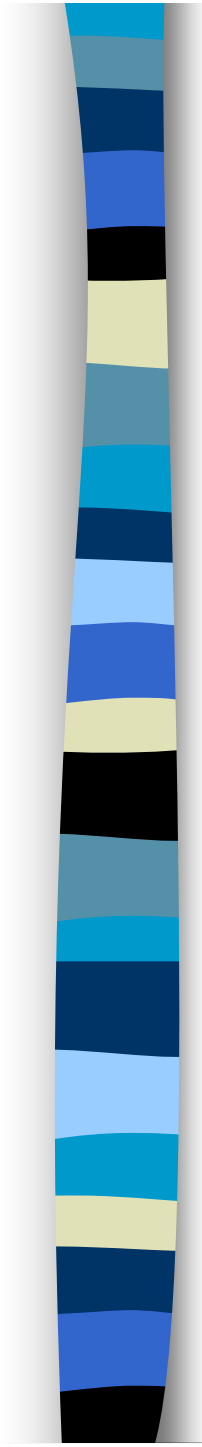
$$P(X = 2) = \binom{6}{2} (0.15)^2 (0.85)^4 = (15)(0.15)^2 (0.85)^4 = 0.17618$$

$$P(X = 3) = \binom{6}{3} (0.15)^3 (0.85)^3 = (20)(0.15)^3 (0.85)^3 = 0.04145$$

$$P(X = 4) = \binom{6}{4} (0.15)^4 (0.85)^2 = (15)(0.15)^4 (0.85)^2 = 0.00549$$

$$P(X = 5) = \binom{6}{5} (0.15)^5 (0.85)^1 = (6)(0.15)^5 (0.85)^1 = 0.00039$$

$$P(X = 6) = \binom{6}{6} (0.15)^6 (0.85)^0 = (1)(0.15)^6 (1)^0 = 0.00001$$



x	$P(X = x)$
0	0.37715
1	0.39933
2	0.17618
3	0.04145
4	0.00549
5	0.00039
6	0.00001

The expected number (mean) of men out of 6 with high blood pressure is:

$$\mu = n\pi = (6)(0.15) = 0.9$$

The variance is:

$$\sigma^2 = n\pi(1 - \pi) = (6)(0.15)(0.85) = 0.765$$



Poisson Distribution:

- It is discrete distribution.
- The discrete r. v. X is said to have a Poisson distribution with parameter (average) λ if the probability distribution of X is given by

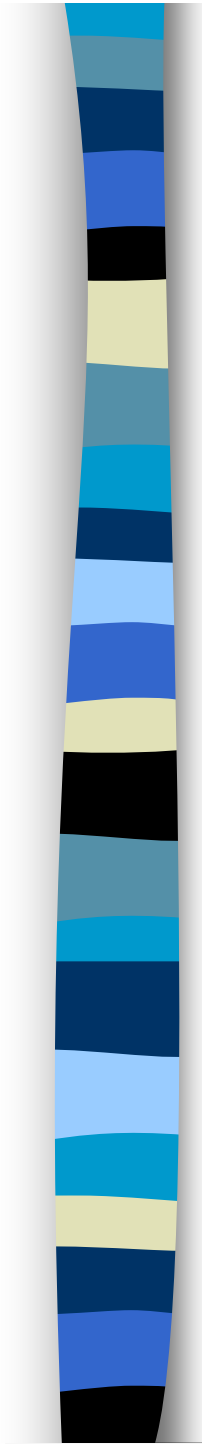
$$P(X = x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & ; \text{ for } x = 0, 1, 2, 3, \dots \\ 0 & ; \text{ otherwise} \end{cases}$$

where $e = 2.71828$ (the natural number $x = e^{\ln x}$).

We write:

$$X \sim \text{Poisson}(\lambda)$$

- The mean (average) of Poisson (λ) is : $\mu = \lambda$
- The variance is: $\sigma^2 = \lambda$



- The Poisson distribution is used to model a discrete r. v. which is a count of how many times a specified random event occurred in an interval of time or space.

Example:

- No. of patients in a waiting room in an hours.
- No. of serious injuries in a particular factory in a month.
- No. of calls received by a telephone operator in a day.
- No. of rates in each house in a particular city.

Note:

λ is the average (mean) of the distribution.

If $X =$ The number of calls received in a month and
 $X \sim$ Poisson (λ)



then:

(i) $Y =$ The no. calls received in a year.

$Y \sim \text{Poisson}(\lambda^*)$, where $\lambda^* = 12\lambda$

$Y \sim \text{Poisson}(12\lambda)$

(ii) $W =$ The no. calls received in a day.

$W \sim \text{Poisson}(\lambda^*)$, where $\lambda^* = \lambda/30$

$W \sim \text{Poisson}(\lambda/30)$

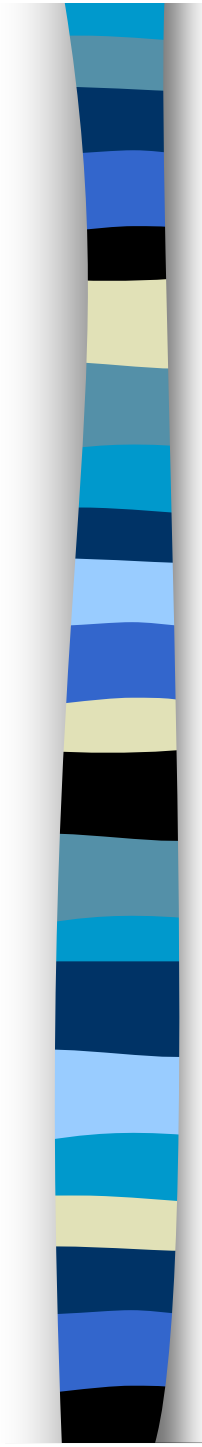
Example:

Suppose that the number of snake bites cases seen at KKUH in a year has a Poisson distribution with average 6 bite cases.

1- What is the probability that in a year:

(i) The no. of snake bite cases will be 7?

(ii) The no. of snake bite cases will be less than 2?



2- What is the probability that in 2 years there will be 10 bite cases?

3- What is the probability that in a month there will be no snake bite cases?

Solution:

(1) $X =$ no. of snake bite cases in a year.

$$X \sim \text{Poisson}(6) \quad (\lambda=6)$$

$$P(X = x) = \frac{e^{-6} 6^x}{x!}; \quad x = 0, 1, 2, \dots$$

(i) $P(X = 7) = \frac{e^{-6} 6^7}{7!} = 0.13768$

(ii) $P(X < 2) = P(X = 0) + P(X = 1)$
 $= \frac{e^{-6} 6^0}{0!} + \frac{e^{-6} 6^1}{1!} = 0.01735$



$Y =$ no of snake bite cases in 2 years

$$Y \sim \text{Poisson}(12) \quad (\lambda^* = 2\lambda = (2)(6) = 12)$$

$$P(Y = y) = \frac{e^{-12} 12^y}{y!} : \quad y = 0, 1, 2, \dots$$

$$\therefore P(Y = 10) = \frac{e^{-12} 12^{10}}{10!} = 0.1048$$

3- $W =$ no. of snake bite cases in a month.

$$W \sim \text{Poisson}(0.5) \quad \lambda^{**} = \frac{\lambda}{12} = \frac{6}{12} = 0.5$$

$$P(W = w) = \frac{e^{-0.5} (0.5)^w}{w!} : \quad w = 0, 1, 2, \dots$$

$$P(W = 0) = \frac{e^{-0.5} (0.5)^0}{0!} = 0.6065$$