Chapter 4:
Probability Distributions

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Some events can be defined using random variables.

Random variables $\begin{cases} \textit{Discrete Random Variables} \\ \\ \textit{Continuous Random Variables} \end{cases}$

4.2. Probability Distributions of Discrete R.V.'s:-

Examples of discrete r v.'s

•The no. of patients visiting KKUH in a week.

•The no. of times a person had a cold in last year.

Example: onsider the following discrete random variable.

X = The number of times a person had a cold in January 1998 in Saudi Arabia.

Suppose we are able to count the no. of people in Saudi Arabia for which $X = x$

x	Frequency of x
(no. of times a person had a cold in January 1998 in S. A.)	(no. of people who had a cold in January 1998 in S.A.)
0	10,000,000
1 2 3	3,000,000 2,000,000 1,000,000
Total	N= 16,000,000

Simple frequency table of no. of times a person had a cold in January 1998 in Saudi Arabia.

Experiment: Selecting a person at random

Define the event:

$(X = x) =$ The event that the selected person had a cold in January 1998 x times.

In particular,

$(X = 2) =$ The event that the selected person had a cold in January 1998 two times.

For this experiment, there are $n(\Omega) = 16{,}000{,}000$ equally likely outcomes.

$$\therefore P(X = x) = \frac{n(X = x)}{n(\Omega)} = \frac{\begin{pmatrix} no.\ of\ \ people\ \ who\ \ had\ \ x \\ colds\ \ in\ \ Januay\ 1998 \end{pmatrix}}{16{,}000{,}000}$$

x	freq. of x $n(X=x)$	$P(X=x)$ $= n(X=x)/1600000$
0	10000000	0.6250
1	3000000	0.1875
2	2000000	0.1250
3	1000000	0.0625
Total	16000000	1.0000

Note:

$$P(X=x)=\frac{n(X=x)}{16000000}$$

$$= \mathrm{Re}\,lative\ Frequency$$

$$=\frac{frequency}{16000000}$$

x	$P(X=x)$
0	0.6250
1	0.1874
2	0.1250
3	0.0625
Total	1.0000

This table is called the probability distribution of the discrete random variable X.

Notes:

- (X=0) , (X=1) , (X=2) , (X=3) are mutually exclusive (disjoint) events.

•$(X=0)$, $(X=1)$, $(X=2)$, $(X=3)$ are mutually exhaustive events

• The probability distribution of any discrete random variable x must satisfy the following two properties:

$$(1) \qquad 0 \le P(X = x) \le 1$$

$$(2) \qquad \sum_{x} P(X = x) = 1$$

•Using the probability distribution of a discrete r.v. we can find the probability of any event expressed in term of the r.v. X.

Example:

Consider the discrete r.v. X in the previous example.

x	$P(X=x)$
0	0.6250
1	0.1875
2	0.1250
3	0.0625

(1) $P(X \geq 2) = P(X = 2) + P(X = 3) = 0.1250 + 0.0625 = 0.1875$

(2) $P(X > 2) = P(X = 3) = 0.0625$

(3) $P(1 \leq X < 3) = P(X = 1) + P(X = 2) = 0.1875 + 0.1250 = 0.3125$

(4) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= 0.6250 + 0.1875 + 0.1250 = 0.9375$$

Or

$$P(X \leq 2) = 1 - P((X \leq 2)^c)$$

$$= 1 - P(X > 2)$$

(5) $P(-1 \leq X < 2) = P(X = 0) + P(X = 1)$

$$= 0.6250 + 0.1875 = 0.8125$$

(6) $P(-1.5 \le X < 1.3) = P(X=0) + P(X=1)$

$$= 0.6250 + 0.1875 = 0.8125$$

(7) $P(X = 3.5) = P(\phi) = 0$

(8) The probability that the selected person had <u>at least</u> 2 colds

$$= P(X \ge 2)$$

$$= P(X=2) + P(X=3) = 0.1875$$

(9) The probability that the selected person had <u>at most</u> 2 colds

$$= P(X \le 2)$$

$$= 0.9375$$

(10) The probability that the selected person had <u>more than</u> 2 colds

$$= P(X > 2)$$

$$= P(X=3) = 0.0625$$

(11) The probability that the selected person had less than 2 colds

$$= P(X < 2)$$
$$= P(X = 0) + P(X = 1) = 0.8125$$

Graphical Presentation:

The probability distribution of a discrete r. v. X can be graphically presented as follows

x	$P(X=x)$
0	0.6250
1	0.1875
2	0.1250
3	0.0625

Population Mean of a Discrete Random

The mean of a discrete random variable X is denoted by μ and defined by:

$$\mu = \sum_x x\,P(X = x) \qquad \text{[mean = expected value]}$$

Example: We wish to calculate the mean μ of the discrete r. v. X in the previous example.

x	$P(X=x)$	$xP(X=x)$
0 1 2 3	0.6250 0.1875 0.1250 0.0625	0.0 0.1875 0.2500 0.1875
Total	$\mu = \sum xP(X = x)$ =0.625	$\sum P(X = x) = 1.00$

$$\mu = \sum_x xP(X = x) = (0)(0.625) + (1)(0.1875) + (2)(0.125) + (3)(0.0625) = 0.625$$

Cumulative Distributions:

The cumulative distribution of a discrete r. v. x is defined by

$$P(X \le x) = \sum_{a \le x} P(X = a)$$

Example: The cumulative distribution of X is the previous example is:

x	$P(X \le x)$	-
0	0.6250	$P(X \le 0) = P(X = 0)$
1	0.8125	$P(X \le 1) = P(X = 0) + P(X = 1)$
2	0.9375	$P(X \le 3) = P(X = 0) + \cdots + P(X = 3)$
3	1.0000	$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

Binomial Distribution:

- It is discrete distribution.
- It is used to model an experiment for which:

1. The experiment has trials.
2. Two possible outcomes for each trial :

   S= success and F= failure
3. (boy or girl, Saudi or non-Saudi,…)
4. The probability of success: is constant for each trial.
5. The trials are independent; that is the outcome of one trial has no effect on the outcome of any other trial

The discrete r. v.

X = The number of successes in the n trials has a binomial distribution with parameter n and π , and we write $X \sim Binomial(n, \pi)$

The probability distribution of X is given by:

$$P(X = x) = \begin{cases} \binom{n}{x} \pi^{x} (1-\pi)^{n-x} & \textit{for } x = 0, 1, 2, \ldots, n \\ 0 & \textit{otherwise} \end{cases}$$

where $$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

We can write the probability distribution of X is a table as follows.

x	$P(X=x)$
0	$\binom{n}{0}\pi^0(1-\pi)^{n-0}=(1-\pi)^n$
1	$\binom{n}{1}\pi^1(1-\pi)^{n-1}$
2	$\binom{n}{2}\pi^2(1-\pi)^{n-2}$
:	:
$n-1$	$\binom{n}{n-1}\pi^{n-1}(1-\pi)^2$
n	$\binom{n}{n}\pi^n(1-\pi)^0=\pi^n$
Total	1

Result:

If $X \sim$ Binomial (n , π) , then

- The mean: $\mu = n\pi$ (expected value)
- The variance: $\sigma^2 = n\pi(1-\pi)$

Example: 4.2 (p.106)

Suppose that the probability that a Saudi man has high blood pressure is 0.15. If we randomly select 6 Saudi men, find the probability distribution of the number of men out of 6 with high blood pressure. Also, find the expected number of men with high blood pressure.

Solution:

X = The number of men with high blood pressure in 6 men.

S = Success: The man has high blood pressure

F = failure: The man does not have high blood pressure.

- Probability of success $P(S) = \pi = 0.15$
- no. of trials $n=6$

$X \sim$ Binomial (6, 0.16)

$$\begin{bmatrix} \pi = 0.15 \\ 1-\pi = 0.85 \\ n = 6 \end{bmatrix}$$

The probability distribution of X is:

$$P(X=x)=\begin{cases} \binom{6}{x}(0.15)^{x}(0.85)^{6-x} & ;x=0,1,2,3,4,5,6 \\ 0 & ; \quad otherwise \end{cases}$$

$$P(X=0)=\binom{6}{0}(0.15)^{0}(0.85)^{6}=(1)(0.15)^{0}(0.85)^{6}=0.37715$$

$$P(X=1)=\binom{6}{1}(0.15)^{1}(0.85)^{5}=(6)(0.151)(0.85)^{5}=0.39933$$

$$P(X=2)=\binom{6}{2}(0.15)^{2}(0.85)^{4}=(15)(0.15)^{2}(0.85)^{4}=0.17618$$

$$P(X=3)=\binom{6}{3}(0.15)^{3}(0.85)^{3}=(20)(0.15)^{3}(0.85)^{3}=0.04145$$

$$\boldsymbol{P}(\boldsymbol{X}=4)=\binom{6}{4}(0.15)^{4}(0.85)^{2}=(15)(0.15)^{4}(0.85)^{2}=0.00549$$

$$P(X=5)=\binom{6}{5}(0.15)^{5}(0.85)^{1}=(6)(0.15)^{5}(0.85)^{1}=0.00039$$

$$P(X=6)=\binom{6}{6}(0.15)^{6}(0.85)^{0}=(1)(0.15)^{6}(1)^{0}=0.00001$$

x	$P(X = x)$
0	0.37715
1	0.39933
2	0.17618
3	0.04145
4	0.00549
5	0.00039
6	0.00001

The expected number (mean) of men out of 6 with high blood pressure is: $\mu = n\pi = (6)(0.15) = 0.9$

The variance is: $\sigma^2 = n\pi(1-\pi) = (6)(0.15)(0.85) = 0.765$

Poisson Distribution:

- It is discrete distribution.
- The discrete r. v. X is said to have a Poisson distribution with parameter (average) λ if the probability distribution of X is given by

$$P(X=x)=\begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & ; \quad for \quad x=0,1,2,3,\ldots \\ 0 & ; \quad otherwise \end{cases}$$

where e = 2.71828 (the natural number $x = e^{\ln x}$).

We write:

$$X \sim \text{Poisson}\ (\ \lambda\ )$$

- The mean (average) of Poisson (λ) is : $\mu = \lambda$
- The variance is: $\sigma^2 = \lambda$

•The Poisson distribution is used to model a discrete r. v. which is a <u>count</u> of how many times a specified random event occurred in an interval of time or space.

Example:

•No. of patients in a waiting room in an hours.

•No. of serious injuries in a particular factory in a month.

•No. of calls received by a telephone operator in a day.

•No. of rates in each house in a particular city.

Note:

λ is the average (mean) of the distribution.

If X = The number of calls received in a month and $X \sim$ Poisson (λ)

then:

(i) Y = The no. calls received in a year.

$Y \sim$ Poisson (λ^*), where $\lambda^*=12\lambda$

$Y \sim$ Poisson (12λ)

(ii) W = The no. calls received in a day.

$W \sim$ Poisson (λ^*), where $\lambda^*=\lambda/30$

$W \sim$ Poisson ($\lambda/30$)

Example:

Suppose that the number of snake bites cases seen at KKUH in a year has a Poisson distribution with average 6 bite cases.

1- What is the probability that in a year:

(i) The no. of snake bite cases will be 7?

(ii) The no. of snake bite cases will be less than 2?

2- What is the probability that <u>in 2 years</u> there will be 10 bite cases?

3- What is the probability that <u>in a month</u> there will be no snake bite cases?

Solution:

(1) $X =$ no. of snake bite cases in <u>a year.</u>

$X \sim$ Poisson (6) (λ=6)

$$P(X = x) = \frac{e^{-6} 6^x}{x!} \; ; \; x = 0, 1, 2, \ldots$$

(i) $$P(X = 7)) = \frac{e^{-6} 6^7}{7!} = 0.13768$$

(ii) $$P(X < 2) = P(X = 0) + P(X = 1)$$

$$= \frac{e^{-6} 6^0}{0!} + \frac{e^{-6} 6^1}{1!} = 0.01735$$

Y = no of snake bite cases in <u>2 years</u>

$$Y \sim \text{Poisson}(12) \qquad (\lambda^{*} = 2\lambda = (2)(6) = 12)$$

$$P(Y = y) = \frac{e^{12}12^{y}}{y!}: \quad y = 0, 1, 2\ldots$$

$$\therefore P(Y = 10) = \frac{e^{-12}12^{10}}{10} = 0.1048$$

3- W = no. of snake bite cases in a <u>month.</u>

$$W \sim \text{Poisson}\ (0.5) \qquad \lambda^{**} = \frac{\lambda}{12} = \frac{6}{12} = 0.5$$

$$\boldsymbol{P(W = w) = \frac{e^{-.5}.5^{w}}{w!}}: \quad \boldsymbol{w} = 0, 1, 2\ldots$$

$$P(W = 0) = \frac{e^{-0.5}(0.5)^{0}}{0!} = 0.6065$$