Testing Used Better than Aged class of life Distributions based on U-Statistics

By

El-Batal, I. [(1)], **Abdel Aziz, A.** [(2)] **and Dewidar, O.**

Abstract

A non-parametric procedure is presented for testing exponentially against used better than aged class of life distributions. Convergence of the proposed statistic to the normal distribution is proved and a function of this process is proposed as a test statistic for large samples. Selected critical values are tabulated for sample size n = 5(1)25(5)50. The Pitman asymptotic relative efficiency to tests of other classes is studied. The power of the test is also estimated by simulation. An example of 40 patients suffering from blood cancer disease demonstrates practical application of the proposed test.

Keywords: U-Statistics, life distributions, aging properties, hypothesis testing, asymptotic normality, efficiency.

(1), (2) The Institute of Statistical Studies and Research – Cairo University.

1- Introduction

Statisticians find it is useful to categorize life distributions according to different aging properties. These categories are useful for modeling situations, maintenance, inventory theory and biometry. Well known classes include IFR, IFRA, NBU, DMRL, NBUE, HNBUE and many other classes.

For definitions and properties, of these classes see, e.g. , Deshpande among others (1986), Hendi and Rady (1994), Barlow and Proschan (1981) and Bryson and Siddiqui (1969).

Studies of testing exponentially against these classes of life distributions have continued over the past three decades or more. Most of the testing procedures are based on developing empirical estimates of measure of departure from exponential in favor of the alternatives.

For the vast literature see, e.g., koul (1978), Kango (1993), Hollander and Proschan (1972), Ahmed (1994) and Deshpande (1983).

Let X be a random variable describing the life time of a brand new device which begins to work at time t = 0. As usual in the reliability literature, we dente by X_t the life time of the device of age t with $t \geq 0$. The probability that the device of age t still working till time x (the survival function) is,

$$\overline{F}_t(x) = p(X > x + t \mid x > t) = \frac{\overline{F}(t+x)}{\overline{F}(t)}$$

is the survival function of x. $\overline{F}(x)$ where

Some properties concerning the asymptotic behavior of X_t as $t \to \infty$ will be used .

1.1 Definition; (Bhattacharjee ,1982),

If X is a nonnegative random variable, its distribution function F(x) is said to be finitely and positively smooth if a number $\gamma \in (0, \infty)$ exists such that

$$\lim_{t \to \infty} \overline{F}_t(x) = e^{(-\gamma x)} \qquad \text{for all } x \geq 0$$

where γ will be called the asymptotic decay coefficient of X.

Denoting by X_e a random variable exponentially distributed by mean $1/\gamma$, the following definition implies that X_t converges to X_e in distribution written as, $X_t \xrightarrow{d} X_e$.

This property is useful for the description of random life times of devices of unknown age .

1.2 Definition :

The random variable X or its distribution function is called used better than aged UBA if,

$$\overline{F}_t(x) \ge e^{-\gamma x} \quad , \text{ for all } t , x \ge 0 \qquad (1\text{-}1)$$

where γ is the asymptotic decay of X .

We observe that the equality of the inequality (1-1) is achieved when F(x) has an exponential distribution with mean μ equal to the coefficient of the asymptotic decay γ , where the exponential distribution is the only one which has the lack of memory property .

This paper is organized as follows : In **Section 2**, a test statistic based on a U-Statistic for testing H_0 : F is exponential vs, H_1 : F is UBA and Not exponential. Monte Carlo null distribution critical pointes are presented for sample size n = 5(1)25(5)50 is investigated in **Section 3**. The Power estimates for sample size n = 5, 10, 20, 25 is presented for many parameters in **section 4**. The Pitman asymptotic efficiency is given for many alternatives in **section 5**. Finally, an example using data from Alwasel (1997) in medical science is given in **section 6**.

2- U-Statistic test procedure for testing UBA class ;

Let $X_1, X_2, \ldots , X_n$ be a random sample taken from a population with distribution
function F which has the property ,

$$\lim_{t\to\infty} \overline{F}_t(x) = e^{-\gamma x} \qquad , x \ge 0 .$$

We wish to test the null hypothesis ,

$H_0 \; : \; \overline{F} = e^{-\mu x}$, i.e. $F \in$ exponential distribution .

against the alternative ,

$H_1 \; : \; F \in$ UBA and not exponential .

From the definition of UBA class of inequality (1-1) under H_0,

$$\overline{F}(t+x) - \overline{F}(t)e^{-\gamma x} = 0 , \qquad \text{for all } t , x \geq 0.$$

while under H_1,

$$\overline{F}(t+x) - \overline{F}(t)e^{-\gamma x} \geq 0.$$

Using the following measure of departure from H_0,

$$\delta_F = E_F\left\{\overline{F}(t+x) - \overline{F}(t)e^{-\gamma x}\right\},$$

$$= \int_0^\infty \int_0^\infty \left\{\overline{F}(t+x) - \overline{F}(t)e^{-\gamma x}\right\} dF(x)\, dF(t),$$

$$\delta_F = \int_0^\infty \int_0^\infty \overline{F}(t+x)\, dF(x)\, dF(t) - \int_0^\infty \overline{F}(t)\, dF(t) \int_0^\infty e^{-\gamma x}\, dF(x) \qquad (2\text{-}1)$$

We observe that , under $H_0 : \delta_F = 0$ and under $H_1 : \delta_F > 0$

Now we want to estimate δ_F by the estimator $\hat{\delta}_F$.

To this end , let $X_1, X_2, \ldots , X_n$ be a random sample from the distribution function F
and $\overline{F}_n(x)$, denote the empirical survival distribution used as an estimator of $\overline{F}(x)$,
then ,

$$\overline{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I\,(x_i > x),$$

where I is the indicator function of $x_i > x$.

Using $\hat{\gamma}$ as an estimator of γ and equal to $\hat{\gamma} = \dfrac{n}{\sum x_i}$, under H_0.

then ,

$$\hat{\delta}_{F_n} = \int_0^\infty \int_0^\infty \overline{F}_n(t+x) dF_n(x)\, dF_n(t) - \int_0^\infty \overline{F}_n(t)\, dF_n(t) \int_0^\infty e^{-\hat{\gamma} x}\, dF_n(x),$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^\infty \int_0^\infty I\ (x_i > t+x\ )\, dF_n(x)\, dF_n(t) - \frac{1}{n} \sum_{i}^{n} I\ (x_i > t)\, dF_n(t) \int_0^\infty e^{-\hat{\gamma} x}\, dF_n(x),$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} I\ (x_i > x_j + x_k) - \frac{1}{n^3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} e^{-\hat{\gamma} x_j} I\ (x_i > x_k),$$

$$\hat{\delta}_{F_n} = \frac{1}{n^3} \sum_i \sum_j \sum_k \left\{ I\ (x_i > x_j + x_k) - e^{-\hat{\gamma} x_j} I\ (x_i > x_k) \right\}. \qquad (2\text{-}2)$$

writes

$$\hat{\delta}_{F_n} = \frac{1}{n^3} \sum_i \sum_j \sum_k \phi\ (x_i, x_j, x_k),$$

where

$$\phi\ (x_i, x_j, x_k) = \left\{ I\ (x_i > x_j + x_k) - e^{-\hat{\gamma} x_j} I\ (x_i > x_k) \right\}$$

Now using the following symmetric Kernel ,

$$\psi\ (x_i, x_j, x_k) = \frac{1}{3!} \sum_R \phi\ (x_i, x_j, x_k).$$

One realizes that , $\hat{\delta}_{F_n}$ Is equivalent to the U_n Statistic given by ,

$$U_n = \frac{1}{\binom{n}{3}} \sum_{i<j<k} \phi\ (x_i, x_j, x_k).$$

The following theorem summarizes the large sample properties of $\hat{\delta}_{F_n}$ or U_n .

2-1 Theorem

(1) As n→ ∞ , the U_n Converges to δ_F with probability one .

(2) As n→ ∞ the $\sqrt{n}(U_n - \delta_F)$ is asymptotically normal with mean zero and Variance σ^2.

(3) Under H_0 the variance reduces to $\sigma_0^2 = \frac{3}{4}$.

(4) If F is continuous UBA then the test is consistent .

Proof

The proof of (1) and (2) comes directly from a theorem of Serfling (1980) .

We next find the variance of U_n, from the discussions in Serfling (1980), the variance is ,

$$V_F\{U_n\} = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c}\binom{n-m}{m-c}\zeta_c,,$$

where in this test , c = 1,2,3 , m = 3 .

Then ,

$$V_F\{U_n\} = \frac{1}{\binom{n}{3}} \sum_{c=1}^{3} \binom{3}{c}\binom{n-3}{3-c}\zeta_c,$$

$$= \frac{6}{n(n-1)(n-2)}\left\{\frac{3}{2}(n-3)(n-4)\zeta_1 + 6(n-3)\zeta_2 + \zeta_3\right\}$$

and for large n ,

$$\sigma^2 = \lim_{n\to\infty} n V_F\{U_n\} = 9\zeta_1,$$

where

$$\zeta_1 = E\{\Phi^2{}_1(X_1)\} - \theta^2,$$

and

$$\Phi_1(x_1) = E\{\Phi\ (x_1, X_2, X_3)\},$$

$$= E\{I\ (x_1 > X_2 + X_3) - e^{-\gamma X_2} I\ (x_1 > X_3)\},$$

$$\Phi_1(x_1) = \int_0^{\infty} \int_0^{x_1 - x_3} dF(x_2)\, dF(x_3) - \int_0^{\infty} e^{-\gamma x_2} dF(x_2) \int_0^{x_1} dF(x_3).$$

Under H_0 : $\bar{F}(x) = e^{-x}$, and $dF(x) = e^{-x} dx$, supposing $\gamma = \mu = 1$, then ,

$$\Phi_1(x_1) = \int_0^{\infty} e^{-x_3} \int_0^{x_1 - x_3} e^{-x_2} dx_2\, dx_3 - \int_0^{\infty} e^{-2x_2} dx_2 \int_0^{x_1} e^{-x_3} dx_3,$$

$$= \int_0^{\infty} e^{-x_3}\left(1 - e^{-(x_1 - x_3)}\right) dx_3 - \frac{1}{2}\left(1 - e^{-x_1}\right) = \frac{1}{2}\left(1 - e^{-x_1}\right),$$

and,

$$(\Phi_1(X_1))^2 = \frac{1}{4} + \frac{1}{4} e^{-2x} - \frac{1}{2} e^{-x} ,$$

$$E\left(\Phi_1^2(X_1)\right) = \frac{1}{4} + \frac{1}{12} - \frac{1}{4} = \frac{1}{12},$$

and

$$\theta^2 = (E\Phi(X_1, X_2, X_3))^2,$$

$$= \int\int\int (I\ (x_1 > x_2 + x_3) - e^{-x_2} I\ (x_1 > x_3))\, dF_{x_1} dF_{x_2} dF_{x_3},$$

then $\theta^2 = \int_0^{\infty} \frac{1}{2}(1 - e^{-x_1}) e^{-x_1} dx_1 = \frac{1}{2} - \frac{1}{2} = 0.$

It follows that

$$\zeta_1 = E\left(\Phi_1^2(X_1)\right) = \frac{1}{12}, \qquad \text{and,} \quad \sigma_0^2 = 9\zeta_1 = \frac{9}{12} = \frac{3}{4}.$$

From the later discussion we find that for n large $\sqrt{n}(U_n - \theta)$ is asymptotically normal with mean zero and variance σ^2 , while under H_0 : $\frac{\sqrt{n}\, U_n}{\sigma_0}$, is asymptotically $(\approx)$normal with mean zero and variance unity ,i.e. $\sqrt{\frac{4}{3}n}\, U_n \approx N(0,1)$, thus the approximate α-level testing H_0 versus H_1, is to reject H_0 in favor of H_1 if $\sqrt{\frac{4}{3}n}\, U_n \geq z_\alpha$.

3- Monte Carlo critical points for $\hat{\delta}_{F_n}$

In practice, simulated percentiles for samples are commonly used by applied statistics and reliability analysts. We have simulated the upper percentile for 90 %, 95 %, 98 % and 99 % by using visual basic programming. Table (3-1) gives these percentile points of the statistic $\hat{\delta}_{F_n}$ in Equation (2-2) and the calculations are based on 5000 simulated samples of size n =5(1)25(5)50, X has Exp(1) and γ =1.

It is clear from Table (3-1) that, the percentiles values increases slowly as the sample size decreases.

Table 3-1

$\hat{\delta}_{F_n}$ Upper percentiles critical values of

n	% 90	% 95	% 98	% 99
5	.0447	.0597	.0754	.0774
6	.0417	.0564	.0728	.0747
7	.0411	.0544	.0711	.0716
8	.0395	.0524	.0668	.0688
9	.0379	.0497	.0631	.0650
10	.0371	.0487	.0622	.0632
11	.0364	.0467	.0608	.0617
12	.0357	.0471	.0579	.0589
13	.0341	.0455	.0570	.0586
14	.0336	.0441	.0557	.0569
15	.0325	.0424	.0526	.0538
16	.0318	.0427	.0528	.0541
17	.0311	.0407	.0513	.0523
18	.0304	.0399	.0501	.0516
19	.0305	.0400	.0500	.0503
20	.0291	.0383	.0476	.0488
21	.0291	.0376	.0475	.0483
22	.0281	.0374	.0477	.0485
23	.0287	.0367	.0449	.0458
24	.0283	.0365	.0457	.0470
25	.0282	.0371	.0462	.0470
30	.0258	.0334	.0416	.0423
35	.0244	.0312	.0401	.0411
40	.0231	.0307	.0382	.0389
45	.0221	.0287	.0349	.0353
50	.0210	.0274	.0340	.0350

4- The power estimates

The power of the test statistic $\hat{\delta}_{F_n}$ is considered for the significance level α = 0.05
and for commonly used distributions in reliability modeling. These distributions are

1- Linear failure rate ; $\overline{F}_1(x) = \text{Exp}(-x - \frac{1}{2}\theta x^2), \quad \theta, x \geq 0$

2- Wiebull ; $\overline{F}_2(x) = \text{Exp}(-x^{\theta}), \quad \theta, x \geq 0.$

Table (4-1) contains the power estimates for $\hat{\delta}_{F_n}$ test statistic with respect to these distributions. The estimates are based on 5000 simulated samples of sizes n = 5, 10, 20, 25 and significance level α = 0.05. We notice clearly the departure from exponentially towards UBA properties as θ increases.

Table 4-1
Power estimates for $\hat{\delta}_{F_n}$

Distribution	parameter	sample size			
	θ	n=5	n=10	n=20	n=25
Linear failure rate	2	.04815	.03505	.02533	.02406
	3	.07272	.06325	.05514	.05408
	4	.08838	.08123	.07429	.07371
Weibull	2	-.0339	-.0286	-.0466	-.0511
	3	-.0546	-.0906	-.1129	-.1186
	4	-.0952	-.1337	-.1550	-.1590

5- Asymptotic relative efficiency

Let T_{1n} , T_{2n} be two test statistics for testing ,

$$H_0 \;:\; F_\theta \in \left\{F_{\theta_n}\right\} ,\theta_n = \theta + c\,n^{\frac{-1}{2}} .$$

where c is an arbitrary constant , then the Pitman asymptotic relative efficiency of T_{1n} Relative to T_{2n} is defined by,

$$\text{Eff}\left(T_{1n} \;,\; T_{2n}\right) = \left(\frac{\mu'_1(\theta_0)}{\mu'_2(\theta_0)}\right)^2 \left(\frac{\sigma_2^2(\theta_0)}{\sigma_1^2(\theta_0)}\right), \tag{5-1}$$

where

$$\mu'_i(\theta_0) = \lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_{in})_{\theta \to \theta_0},$$

and

$$\sigma_i^2(\theta_0) = \lim_{n\to 0} \text{Var}(T_{in}) \qquad ,i = 1,2.$$

We choose the following three alternatives ,

1- The linear failure rate distribution given by ,

$$\overline{F}_1(x;\theta) = e^{-\left(x + \frac{\theta}{2}x^2\right)}, \quad \theta \ge 0, \quad x \ge 0. \tag{5-2}$$

2- The Makehame distribution given by ,

$$\overline{F}_2(x;\theta) = e^{-\left(x + \theta(x + e^{-x} - 1)\right)}, \; \theta, x \ge 0. \tag{5-3}$$

3- The Weibull distribution given by ,

$$\overline{F}_3(x;\theta) = e^{-x^\theta} , \quad \theta, x \ge 0. \tag{5-4}$$

First getting the efficiency for linear failure rate distribution ,we have , $\overline{F}_1(x) = e^{-\left(x + \frac{\theta}{2}x^2\right)}$, then by differentiation we find ,

$$f_1(x) = (1 + \theta\, x)\, e^{-\left(x + \frac{\theta}{2}x^2\right)} \text{ ,and } dF_1(x) = f_1(x)dx.$$

Using (2-1) and substituting using linear failure rate distribution in (5-2),

$$\frac{\partial}{\partial\theta}\partial F_{\theta}\Big\downarrow_{\theta=0} = \frac{\partial}{\partial\theta}\left\{\int_0^\infty\int_0^\infty \overline{F}_\theta(t+x)\,dF_\theta(x)\,dF_\theta(t) - \int_0^\infty \overline{F}_\theta(t)\,dF_\theta(t)\int_0^\infty e^{-\gamma x}\;dF_\theta(x)\right\}.$$

(5-5)

The first term,

$$\frac{\partial}{\partial\theta}\int_0^\infty\int_0^\infty \overline{F}_\theta(t+x)\;dF_\theta(x)\;dF_\theta(t)$$

$$= \frac{\partial}{\partial\theta}\int_0^\infty\int_0^\infty e^{-(t+x+\frac{\theta}{2}(t+x)^2)}(1+\theta x)e^{-(x+\frac{\theta}{2}x^2)}(1+\theta t)e^{-(t+\frac{\theta}{2}t^2)}dx\,dt,$$

(5-6)

$$= \frac{\partial}{\partial\theta}\int_0^\infty(1+\theta t)e^{-2(t+\frac{\theta}{2}t^2)}\int_0^\infty(1+\theta x)e^{-2(x+\frac{\theta}{2}x^2+\frac{\theta}{2}tx)}\,dx\,dt = -\frac{3}{16}.$$

The second term,

$$-\frac{\partial}{\partial\theta}\int_0^\infty \overline{F}_\theta(t)\,dF_\theta(t)\int_0^\infty e^{-\gamma x}\,dF_\theta(x),$$

$$= -\frac{\partial}{\partial\theta}\int_0^\infty(1+\theta t)e^{-2(t+\frac{\theta}{2}t^2)}dt\int_0^\infty e^{-x}\,(1+\theta x)e^{-(x+\frac{\theta}{2}x^2)}dx,$$

$$= -\frac{\partial}{\partial\theta}\,\frac{1}{2}\int_0^\infty(1+\theta x)e^{-(2x+\frac{\theta}{2}x^2)} = 0 \qquad (5\text{-}7)$$

Summing the two terms in (5-6) and (5-7) the result is $\mu'_1(0) = -\frac{3}{16}$.

then the efficiency for the linear failure rate is,

$$\text{Eff }(\partial_{F_1}) = \left(\frac{\mu'_1(\theta_0)}{\sigma_1(\theta_0)}\right)^2 = \frac{(-\frac{3}{16})^2}{\frac{9}{12}} = .0468 \qquad (5\text{-}8)$$

Second getting the efficiency for Makeham distribution.

By differentiation the Makeham distribution with respect to x in (4-7) we have,

$$f_2(x)=(1+\theta(1-e^{-x})e^{-(x+\theta(x+e^{-x}-1))}, \text{and} \quad dF_2(x)=f_2(x)dx.$$

Using the equation (5-5),
the first term,

$$\frac{\partial}{\partial\theta}\int_0^\infty\int_0^\infty \overline{F}_\theta(t+x)\ dF_\theta(x)\ dF_\theta(t)$$

$$=\frac{\partial}{\partial\theta}\int_0^\infty\int_0^\infty e^{-((t+x)+\theta(t+x+e^{-(t+x)}-1))}(1+\theta(1-e^{-x}))\,\ell^{-(x+\theta(x+e^{-x}-1))}(1+\theta(1-e^{-t}))\,e^{-(t+\theta(t+e^{-t}-1)}dx\,dt$$

$$=\frac{\partial}{\partial\theta}\int_0^\infty(1+\theta(1-e^{-t}))e^{-(2t+\theta(2t+e^{-t}-1))}\int_0^\infty(1+\theta(1-e^{-x}))e^{-(2x+\theta(2x+e^{-x}(1+e^{-t})-2))}dx\,dt,$$

using 1=(1+θ)-θ ,then ,

$$\frac{\partial}{\partial\theta}(1+\theta)\int_0^\infty\frac{-1}{2+\theta(1-e^{-t})}e^{-2(t+\theta(t+e^{-t}-1))}dt-\frac{\partial}{\partial\theta}\theta\int_0^\infty\frac{-2}{3+\theta(1-e^{-t})}e^{-2(t+\theta(t+e^{-t}-1))}dt,$$

$$=\int_0^\infty\left(-\frac{3}{4}e^{-2t}+\frac{1}{4}e^{-3t}\right)dt=-\frac{7}{24}. \qquad (5\text{-}9)$$

Th e second term,

$$-\frac{\partial}{\partial\theta}\int_0^\infty\overline{F}_\theta(t)dF_\theta(t)\int_0^\infty e^{-\gamma x}\ dF_\theta(x),$$

$$=-\frac{\partial}{\partial\theta}\int_0^\infty e^{-2(t+\theta(t+e^{-t}-1))}(1+\theta(1-e^{-t}-1))dt\int_0^\infty(1+\theta(1-e^{-x}))e^{-(2x+\theta(x+\ell^{-x}-1))}dx,$$

$$=\frac{\partial}{\partial\theta}\left\{-\frac{1}{2}\int_0^\infty(2+\theta(1-e^{-x}))e^{-(2x+\theta(x+e^{-x}-1))}dx+\frac{1}{2}\int_0^\infty e^{-(2x+\theta(x+e^{-x}-1))}dx\right\}=0$$

(5-10)

Summing the two terms in (5-9) and (5-10), the result is, $\mu'_2(0)=-\frac{7}{24}.$

then the efficiency for the Makeham distribution is,

$$\textbf{Eff}\left(\partial_{F_2}\right)=\left(\frac{\mu'_2(\theta_0)}{\sigma_2(\theta_0)}\right)^2=\frac{(\frac{-7}{24})^2}{\frac{9}{12}}=.1134 \qquad (5\text{-}11)$$

6- An application

In this section we calculate the test statistic $\hat{\delta}_{F_n}$ for the data representing the life times of 40 patients suffering from blood cancer (Leukemia) at one of the hospitals of the ministry of health in Saudi Arabia (see Alwasel (1997)). The ordered life times (in days) are,

115, 181, 225, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

Using Equation (2-2), we find that the value of the statistic $\hat{\delta}_{F_n}$ for this set of data is 0.0891.

It is clear from the computed value of the test statistic that we accept H_1 which states that this set of data has UBA property under all stated significant levels.

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