

Statistics 581, Problem Set 1

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Reading: Lehmann & Casella, TPE, pages 1 - 32; skim Chapter 0 handout; read Chapter 1 handout.

Due: Wednesday, October 9, 2002.

1. Let X and Y be i.i.d. $\text{Uniform}(0, 1)$ random variables. Define $U = X - Y$, $V = \max(X, Y) = X \vee Y$.
 - (i) What is the range of (U, V) ?
 - (ii) Find the joint density function $f_{U,V}(u, v)$ of the pair (U, V) . Are U and V independent?
2. Ferguson, ACILST, #6, page 7. (This is known as the Polya-Cantelli lemma; see Chapter 2, Proposition 2.11, page 10.)
3. Suppose that for $\theta \in R$,

$$f_{\theta}(u, v) = \{1 + \theta(1 - 2u)(1 - 2v)\}1_{[0,1]^2}(u, v).$$

- (a) For what values of θ is f_{θ} a density function on $[0, 1]^2$?
 - (b) For the set of θ 's you identified in (a), find the corresponding distribution function F_{θ} and show that it has $\text{Uniform}(0, 1)$ marginal distributions.
 - (c) If $(U, V) \sim F_{\theta}$, compute the correlation $\rho(U, V) \equiv \rho$ as a function of θ . Does this show any difficulty with this family of distributions as a model of dependence?
4. Suppose that F is the distribution function of random variables X, Y with $X \sim \text{Uniform}(0, 1)$ marginally and $Y \sim \text{Uniform}(0, 1)$ marginally. Thus $F(x, y) = P(X \leq x, Y \leq y)$ satisfies

$$F(x, 1) = x, \quad 0 \leq x \leq 1, \quad \text{and} \quad F(1, y) = y, \quad 0 \leq y \leq 1.$$

- (a) Show that

$$F(x, y) \leq x \wedge y \equiv F_U(x, y)$$

for all $0 \leq x \leq 1, 0 \leq y \leq 1$. Here $x \wedge y \equiv x$ if $x \leq y, y$ if $y \leq x$.

(b) Show that

$$F(x, y) \geq (x + y - 1)^+ \equiv F_L(x, y)$$

for all $0 \leq x \leq 1, 0 \leq y \leq 1$. Here $z^+ = z1_{[0, \infty)}(z)$.

(c) Show that F_U is the distribution function of (X, X) where $X \sim \text{Uniform}(0, 1)$. Show that F_L is the distribution function of $(X, 1 - X)$ where $X \sim \text{Uniform}(0, 1)$.

(d) The distribution functions F_U and F_L are called the Fréchet bounds. Show that F_L and F_U are singular with respect to Lebesgue measure λ_2 on $[0, 1]^2$; i.e. show that the corresponding probability measures P_L and P_U satisfy

$$P((X, Y) \in A) = 1, \quad \lambda_2(A) = 0$$

and

$$P((X, Y) \in A^c) = 0, \quad \lambda_2(A^c) = 1$$

for some set A (which will be different for P_L and P_U). This implies that F_L and F_U do not have densities with respect to Lebesgue measure on $[0, 1]^2$. (See Chapter 0, Section 3, especially Definition 3.1 and Theorem 3.1.)

5. Lehmann and Casella, TPE, problem 1.10, page 62.