Statistics 581, Problem Set 2

Wellner; 10/9/2002

Reading: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 1-25.
Due: Wednesday, October 16, 2002.

1. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that
$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}\,;$$
i.e.
$$E(Y - EY)^2 = E\{E[(Y - E(Y|X))^2|X]\} + E\{[E(Y|X) - E(Y)]^2\}\,.$$
(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi^2_n(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).
(d) Show that
$$\frac{\chi^2_n(\delta) - (n+\delta)}{\sqrt{2n + 4\delta}} \rightarrow_d N(0,1)$$
as either $n \to \infty$ or $\delta \to \infty$.

2. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi^2_k(\delta)$ with $\delta = \mu'\mu$.

3. (a) Ferguson, ACILST, #4, page 6:
Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.
(b) Give an example of a sequence of random variables X_n for which $X_n \rightarrow_p 0$ but $X_n \rightarrow_{a.s.} 0$ fails.

4. (a) If $W \sim \chi^2_2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W, distribution function F_W, and inverse distribution function F_W^{-1} explicitly.

(b) Suppose that $(X,Y) \sim N_2(0,I)$. Show that R and Θ defined by $R^2 = X^2+Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi^2_2$ and $\Theta \sim \text{Uniform}(0,2\pi)$.
(c) Use the results of (a) and (b) to show (using Theorem 2.3.4) how to use two independent Uniform$(0,1)$ random variables U and V to generate two standard normal random variables.

5. Suppose that $U \sim \text{Uniform}(0,1)$, $\alpha > 0$, and

$$X_n \equiv (n^\alpha / \log(n+1)) 1_{[0,1/n^\alpha]}(U)\,.$$

(a) Show that $X_n \to_{a.s.} 0$ and $E(X_n) \to E(0) = 0$.

(b) Can you find a random variable Y with $|X_n| \le Y$ for all n with $E(Y) < \infty$ for any α?
(c) For what values of α does the uniform integrability condition

$$\limsup_{n\to\infty} E\{|X_n| 1_{[|X_n|\ge M]}\} \to 0 \qquad \text{as} \quad M \to \infty$$

hold?

6. **Optional Bonus Problem 1:** (a) Lehmann and Casella, #3.5, page 64.
(b) Lehmann and Casella, #3.6, page 64.
(c) Lehmann and Casella, #3.7, page 64.

7. **Optional Bonus Problem 2:** Suppose that $X \sim F$ on $R^+ \equiv [0,\infty)$, $Y \sim G$ on R^+, and X and Y are independent random variables. Let $Z = \min\{X,Y\} = X \wedge Y$ and $\Delta = 1\{X \le Y\}$. (This is *right-censored data*: if we view X as a survival time, and Y as a censoring time, then $Z = X$ when $X \le Y$, but $Z = Y$ when $X > Y$.)
(a) Find the joint distribution of (Z,Δ).
(b) If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, show that Z and Δ are independent.
[Hint: for (a), compute $P(Z \le z, \Delta = 1)$ and $P(Z \le z, \Delta = 0)$.]