



An aging notion derived from the increasing convex ordering: the NBUCA class

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Abstract

In this paper, comparison between the life distribution of a new unit with that of the remaining life or a used unit in the increasing convex order leads us to introduce a new class of life distributions which we call new better than used in increasing convex average order and denote by NBUCA. This class includes as subclasses the new better than used, NBU and the new better than used in increasing convex ordering, NBUC. Several properties of this class are presented, including the preservation under convolution, random maxima, mixing and formation of coherent structures. Stochastic comparisons of the excess lifetime at different times of a renewal process when the interarrival times belong to the NBUCA class are established. We provide also some applications of Poisson shock models. Finally testing exponentially against NBUCA is presented.

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1. Motivation and definitions

In the context of lifetime distributions, some orderings of distributions have been used to give characterizations and new definitions of aging classes. By aging we mean the

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phenomenon whereby an older system has a shorter remaining lifetime, in some statistical sense than a younger one (Bryson and Siddiqui, 1969). One of the most important approaches to the study of aging is based on the concept of additional residual life. Let X be a lifetime random variable such that distribution function F with $F(0) = 0$. Given a unit which has survived up to time t , its *additional residual life* (Barlow and Proschan, 1981) is given by

$$X_t = [X - t | X > t], \quad t \in (\alpha, \beta),$$

where $\alpha = \sup\{x : F_X(x) = 0\}$ and $\beta = \inf\{x : F_X(x) = 1\}$.

The comparison of the additional residual life at different times has been used to produce several notions of aging. Below, we present some of the stochastic ordering and aging classes considered in this paper (see Shaked and Shanthikumar, 1994 for a general reference).

Definition 1.1. Let X and Y two non-negative random variables with distribution functions $F(x)$ and $G(x)$, and survival functions $\bar{F}(x)$ and $\bar{G}(x)$, respectively. X is said to be smaller than Y in the

- (i) usual stochastic order, denoted by $X \leq_{st} Y$, if

$$\bar{F}(x) \leq \bar{G}(x) \quad \text{for all } x;$$

- (ii) increasing convex order, denoted by $X \leq_{icx} Y$, if

$$\int_x^\infty \bar{F}(u) du \leq \int_x^\infty \bar{G}(u) du \quad \text{for all } x;$$

- (iii) increasing convex average order, denoted by $X \leq_{icxa} Y$, if

$$\int_0^\infty \int_x^\infty \bar{F}(u) du dx \leq \int_0^\infty \int_x^\infty \bar{G}(u) du dx \quad \text{for all } x. \quad (1.1)$$

In the economics theory, the above orders are respectively known as *first-order stochastic dominance* denoted by X FSD Y , *second-order stochastic dominance* denoted by X SSD Y , and *weak third-order stochastic dominance* denoted by X WTSD Y (for more details, see Deshpande et al., 1986; Kaur et al., 1994).

Definition 1.2. A random variable X or F is said to be:

- (i) new better than used, denoted by $X \in \text{NBU}$, if

$$X_t \leq_{st} X \quad \text{for all } t \geq 0;$$

- (ii) new better than used in the convex order, denoted by $X \in \text{NBUC}$, if

$$X_t \leq_{icx} X \quad \text{for all } t \geq 0;$$

- (iii) new better than used in expectation, denoted by $X \in \text{NBUE}$, if

$$E(X_t) \leq E(X) \quad \text{for all } t \geq 0.$$

The NBU class, introduced by Bryson and Siddiqui (1969) and independently by Marshall and Proschan (1972), has grown to become one of the most studied classes of life distributions. Several extensions of the NBU class have been proposed in the literature. Cao and Wang (1991) suggested the NBUC class while Marshall and Proschan (1972) introduced the NBUE class. The classes NBU, NBUC and NBUE have proved to be very useful in performing analyses of life lengths as well as usable in many replacement policies hence a lot of related results have been obtained in the literature (see, e.g., Klar and Muller, 2003; Utkin and Gurov, 2002; Barlow, 1999).

The NBUE class, however, does not correspond to any of the orderings in Definition 1.1. The NBU class corresponds to Part (i) of Definition 1.1 while the NBUC class corresponds to Part (ii). Hence another class corresponding to Part (iii) of the definition is needed and that is what we do here. Thus the new class we develop and call “new better than used in increasing convex average denoted by NBUCA” is an extension of the NBU and NBUC classes and is different from the NBUE class. Precisely, we have:

Definition 1.3. A random variable X or F is said to be new better than used in the increasing convex average order denoted by NBUCA if

$$X_t \leq_{\text{icxa}} X,$$

equivalently, $X \in \text{NBUCA}$ iff

$$\int_0^\infty \int_x^\infty \bar{F}(u+t) \, du \, dx \leq \bar{F}(t) \int_0^\infty \int_x^\infty \bar{F}(u) \, du \, dx \quad \text{for all } t \geq 0.$$

The above class expands the NBUC class to much bigger one but in a direction different than that of the NBUE class. There are no implications between our new class and that of the NBUE. Its dual class is new worse than used in increasing convex average ordering, denoted by NWUCA, which is defined by reversing the above inequality.

It is obvious that, $X \in \text{NBUCA}$ iff

$$X \text{ WTSD } X_t \text{ for all } t \geq 0$$

or

$$2 \int_0^\infty \bar{v}(x+t) \, dx \leq \mu_2 \bar{F}(t), \tag{1.2}$$

where $\mu_2 = E(X^2)$, assumed finite, and $\bar{v}(x) = \int_x^\infty \bar{F}(u) \, du$.

To introduce a discrete version of this class define $N = \{0, 1, \dots\}$ and let $\{P_k, k \in N\}$ be discrete distribution and define $\bar{P}_k = 1 - P_k$ and $p_k = \bar{P}_k - \bar{P}_{k+1}$, where $k \geq 0$. Suppose also that $\bar{P}_0 = 1$. Observe that our definition of p_k is different from that of Esary et al. (1973), and corresponds to that of Abouammoh and Ahmed (1988).

Definition 1.4. A discrete distribution $\{P_k, k \in N\}$, is said to be discrete NBUCA if,

$$\sum_{i=0}^\infty \sum_{j=k+i}^\infty \bar{P}_j \leq \bar{P}_k \sum_{i=0}^\infty \sum_{j=i}^\infty \bar{P}_j \quad \text{for all } k \geq 0. \tag{1.3}$$

The paper is organized as follows: In Section 2, several properties of this class are presented, including the preservation under convolution, random maxima, mixing and formation of coherent structures. In that section, we give some reversed preservation properties of this class. In Section 3, we add a new result when the stochastic comparisons are given in terms of increasing convex average. Some applications of Poisson shock models are given in Section 4. Finally in Section 5, we address the question of testing $H_0 : F$ is exponential against $H_1 : F \in \text{NBUCA}$ and not exponential. To allow for an easy flow of the material, all proofs are deferred to an appendix.

2. Preservation properties

Useful properties of aging classes of life distributions are the closure with respect to typical reliability operations, see, e.g., Barlow and Proschan (1981). In this section we present some preservation results for the NBUCA class.

2.1. Convolution

As an important reliability operation, convolutions of life distributions of a certain class is often paid much attention. The closure properties of NBUC class were pointed out in Cao and Wang (1991) and Hu and Xie (2002). In the next theorem we establish the closure property of the NBUCA class under the convolution operation.

Theorem 2.1. *Suppose that F_1 and F_2 are two independent NBUCA life distributions. Then their convolution is also NBUCA.*

2.2. Random minima

Preservation properties of some stochastic orders and aging classes as well as their duals under random minima and maxima have been studied by several authors. For more details, one may refer to Shaked (1975), Bartoszewicz (2001) and Li and Zuo (2004). In the following, we give a preservation result for the NBUCA class under formation of random maxima. First, we give the following preliminary result.

Theorem 2.2. *Let X_1, X_2, \dots and Y_1, Y_2, \dots each be a sequence of i.i.d. random variables, and N is independent of X_i 's and Y_i 's. If X_i 's and Y_i 's are both non-negative and with common left end point 0, then $X_i \leq_{\text{icxa}} Y_i$ for $i = 1, 2, \dots$, implies*

$$\max\{X_1, \dots, X_N\} \leq_{\text{icxa}} \max\{Y_1, \dots, Y_N\}.$$

Theorem 2.3. *Let X_1, X_2, \dots be a sequence of i.i.d. random lives, and N is independent of X_i 's. If X_1 is NBUCA, then $\max\{X_1, \dots, X_N\}$ is also of NBUCA property.*

On the other hand, it is a well known fact that some aging notions are preserved under formation of parallel and/or series system (see Barlow and Proschan, 1981; Abouammoh

and El-Newehi, 1986; Hendi and Mashhour, 1993; Li and Kochar, 2001; Pellerey and Petakos, 2002). As a consequence to Theorem 2.3, Corollary 2.1 below is immediate.

Corollary 2.1. *Let X_1, X_2, \dots, X_n be a set of NBUCA independent identically distributed components and consider $T_n = \max\{X_1, X_2, \dots, X_n\}$. Then $T_n \in \text{NBUCA}$.*

2.3. Reversed preservation properties

Recently, Li and Yam (2004) developed some reversed preservation of some stochastic orders and aging conceptions for the parallel (series) systems which are composed of independent and identical elements (i.i.d). Next, we will make a discussion on the reversed preservation property of the NWUCA aging class of life distributions.

Theorem 2.4. *Assume that X_1, \dots, X_n are i.i.d copies of X . For any integer $n > 1$, if $\min\{X_1, \dots, X_n\}$ is NWUCA, then X is also NWUCA.*

2.4. Mixture

The NBUCA class is not closed under mixtures, since mixtures of some exponential life distributions often belong to the DFR class (Barlow and Proschan, 1981).

3. Stochastic comparisons of excess lifetimes of renewal processes

Let us consider a renewal process with independent and identically distributed non-negative interarrival times X_i with common distribution F and $F(0) = 0$. Let $S_0 = 0$ and $S_k = \sum_{i=1}^k X_i$ and consider the renewal counting process $N(t) = \text{Sup}\{n : S_n \leq t\}$. Several papers have investigated some characteristics of the renewal process related to aging properties of F (see, for instance, Barlow and Proschan, 1981; Shaked and Zhu, 1992, Brown, 1980,1981).

Chen (1994) investigated the relationship between the behavior of the renewal function $M(t) = E(N(t))$ and aging property of F . Some other results are given for the excess lifetime at time $t \geq 0$, that is, $\gamma(t) = S_{n(t)+1} - t$, which is the time of the next event a time t . Some examples of such results are the following:

(a) Chen (1994) showed that

- (i) if $\gamma(t)$ is stochastically decreasing in $t \geq 0$, then $F \in \text{NBU}$ and
- (ii) if $E[\gamma(t)]$ is stochastically decreasing in $t \geq 0$, then $F \in \text{NBUE}$;

(b) Li et al. (2000) showed that, if $\gamma(t)$ is decreasing in $t \geq 0$ in the increasing convex order then $F \in \text{NBUC}$.

Whereas these results give sufficient conditions for the aging property of F , in practical situations it would be more interesting to derive some properties for $\gamma(t)$ from the ageing property of F . In fact, given a renewal process it is more feasible to study it if F has some

ageing property than if $\gamma(t)$ has some of the previous properties. A result in such a direction is the following one:

- (a) Barlow and Proschan (1981, p. 169) showed that, if $F \in \text{NBU}$, then $\gamma(t) \leq_{\text{st}} \gamma(0)$, for all $t \geq 0$.
- (b) Belzunce et al. (2001) showed that:
 - (i) if $F \in \text{NBUE}$, then $E[\gamma(t)] \leq E[\gamma(0)]$, for all $t \geq 0$;
 - (ii) if $F \in \text{NBUC}$, then $\gamma(t) \leq_{\text{icx}} \gamma(0)$, for all $t \geq 0$.

The following two theorems present the parallel results for the NBUCA class. Since the two results can be proved in a similar way to Theorem 7 of Li et al. (2000) and Theorem 2.2 of Belzunce et al. (2001), respectively, the proofs are omitted.

Theorem 3.1. *If $\gamma(t)$ is decreasing in t for all $t \geq 0$ in the increasing convex average order then $F \in \text{NBUCA}$.*

Theorem 3.2. *Given a renewal process as above, if $F \in \text{NBUCA}$, then*

$$E(\gamma^2(t)) \leq E(\gamma^2(0)), \quad \text{for all } t \geq 0.$$

4. Application to shock models

Shock models are of great interest in the context of reliability theory. Suppose that a device is subjected to shocks occurring randomly as events in a Poisson process with constant λ . Suppose further that the device has probability \bar{P}_k of surviving the first k shocks. Then the survival function of the device is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{e^{-\lambda t} (\lambda t)^k}{k!}, \quad t \geq 0. \quad (4.1)$$

For the discrete distribution $\{P_k, k \in N\}$, it is well known that the properties of P_k are reflected in corresponding properties of the continuous life distribution $H(t)$. For more details see for instance, Esary et al. (1973), Klefsjo (1981, 1983) and Fagioli and Pellerey (1993). Next, we give a similar result for the NBUCA class.

Theorem 4.1. *The survival function $\bar{H}(t)$ in (4.1) is NBUCA if and only if $\{P_k, k \in N\}$ is discrete NBUCA.*

Let $N(t)$ be the number of shocks interval $(0, t]$. The k th shock arrives at time T_k , $U_{k+1} = T_{k+1} - T_k$, $k = 0, 1, \dots$. We assume that U_1, U_2, \dots are mutually independent and $\lim_{n \rightarrow \infty} T_n = +\infty$ with probability 1. Let

$$a_k(t) = P[N(t) = k, k = 0, 1, \dots,]$$

and

$$A_k(x) = \int_x^\infty a_k(t) dt.$$

Proposition 4.1. $H \in$ NBUCA if for any k and x the following conditions hold,

- (a) $\bar{P}_k \sum_{j=0}^\infty \bar{P}_j \int_0^\infty A_j(x) dx \geq \sum_{j=k}^\infty \bar{P}_j \int_0^\infty A_j(x) dx$, and
- (b) $A_{kj}(x) \sum_{j=0}^k a_j(t) \geq A_k(x + t)$.

Theorem 4.2. Let $U_k \in$ NBUCA, $\int_0^\infty A_k(x) dx$ decreasing and bounded in k . If \bar{P}_k is discrete NBUCA, then the Condition (a) holds.

5. Testing in the NBUCA class

In the context of reliability and life testing, the hazard rate of a life distribution plays an important role for stochastic modeling and classification. Being a ratio of probability density function and the corresponding survival function, it uniquely determines the underlying distribution and exhibits different monotonic behaviors. The concept of the *ageless* notion is equivalent to the phenomenon that age has no effect on the hazard rate. Thus the *ageless* property is equal to constant hazard rate, that is, the distribution is exponential. Hence testing non-parametric classes is done by testing exponentially versus some kind of classes. This applies to many non-parametric classes such as the NBU, NBUC and NBUE, among others.

For a recent literature on testing the above classes as well as others we refer the readers to Ahmad (2001), Ahmad and Mugdadi (2004) and Kayid and Ahmad (2004). Much of the earlier literature is cited in those papers where definitions, inter-relations and discussion of above classes and others are presented.

The problem we propose in this section is that we want to test $H_0 : F$ in exponential against $H_1 : F$ is NBUCA and not exponential. The following result gives a measure of departure from H_0 in favor of H_1 .

Theorem 5.1. Suppose that F is NBUCA life distribution such that its μ_{r+3} the $(r + 3)$ rd moment is finite for some integer $r \geq 0$, then

$$2\mu_{r+3} \leq (r + 2)(r + 3)\mu_2\mu_{r+1}.$$

According to the above theorem, we set the following measure of departure from H_0 :

$$\delta(r) = (r + 2)(r + 3)\mu_2\mu_{r+1} - 2\mu_{r+3}.$$

Note that $\delta(r) = 0$ under H_0 and is positive for H_1 . To make the test scale invariant, we adjust it by μ^{r+3} and define the measure

$$\Delta(r) = \frac{\delta(r)}{\mu^{r+3}},$$

where $\mu = \mu_1$.

To estimate $\Delta(r)$, let X_1, \dots, X_n be a random sample from the life distribution F . Let $\bar{X} = 1/n \sum_{i=1}^n X_i$ be the sample mean. We estimate $\Delta(r)$ by

$$\hat{\Delta}(r) = \frac{1}{\bar{X}^{r+3}n(n-1)} \sum_{i \neq j} \{(r+2)(r+3)X_i^2 X_j^{r+1} - 2X_i^{r+3}\}.$$

Theorem 5.2. As $n \rightarrow \infty$, $\sqrt{n}(\hat{\Delta}(r) - \Delta(r))$ is asymptotically normal with mean zero and variance $\tau^2(r)$ given in (A.5). Under H_0 , the variance is given by (A.6).

To carry out this test, we calculate $n^{1/2}\hat{\Delta}(r)/\tau_0(r)$ and reject if larger than Z_α , the α th normal variate. The choice of r is a question that needs to be addressed. There are three possible routs. Either to choose a small value like $r=0$ or $r=1$ to make calculations simple or to try to find the value of r that gives the maximum power or efficiency if we have some belief about an alternative. To choose r which maximizes the power one can use empirical calculations by simulating sampling from an alternative we believe in and calculating the empirical powers for various sample sizes at $r=0, 1, \dots$ etc. and choose the r that gives the best power.

However, choosing the r that maximizes the efficiency is easy to do theoretically if we use the Pitman notion of efficiency. Let us consider two alternatives that are commonly used and are in the NBUCA class:

(i) *The linear failure rate:*

$$\bar{F}_\Theta^{(1)}(x) = e^{-x - \frac{\theta}{2}x^2}, \quad x \geq 0, \theta \geq 0;$$

(ii) *The Makehaun:*

$$\bar{F}_\Theta^{(2)}(x) = e^{-x - \theta(x + e^{-x} - 1)}, \quad x \geq 0, \theta \geq 0.$$

Note that the null is at $\theta=0$ for these alternatives. The Pitman asymptotic efficacy of the test based on $\Delta(r)$ is given by

$$\begin{aligned} \text{PAE}(\Delta_\theta(r)) &= \frac{|\frac{1}{d\theta}\Delta_\theta(r)|_{\theta \rightarrow \theta_0}}{\tau_0(r)} \\ &= \frac{|(r+2)(r+3)(2\mu'_{r+1, \theta_0} + (r+1)!\mu'_{2, \theta_0}) - 2\mu'_{r+3, \theta_0}|}{\tau_0(r)}, \end{aligned}$$

where $\mu'_{r, \theta_0} = |\frac{d}{d\theta}\mu_{r, \theta}|_{\theta \rightarrow \theta_0}$.

Working with the above two alternatives we get:

$$\text{PAE (Linear failure rate)} = \frac{4(r+1)(r+3)!}{\tau_0(r)} \quad (5.1)$$

and

$$\text{PAE (Makehaun)} = \frac{[7 - 3(\frac{1}{2})^{r+1}](r+3)!}{2\tau_0(r)}. \quad (5.2)$$

Both (5.1) and (5.2) are decreasing in r . Thus $r=0$ is what we choose here.

In this case the PAEs are respectively equal to 0.894 and 0.615. The corresponding values of the NBUE test of [Hollander and Proschan \(1975\)](#) are 0.866 and 0.289. Clearly our test is better and also much simpler to do. [Ahmad and Mugdadi \(2004\)](#) gave a test for the NBUC class which is smaller than one and reported PAEs values of 0.894 and 0.167. Thus our test is better and works for a much larger class, the NBUCA.

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Appendix A.

Proof of Theorem 2.1. The survival function of the convolution of two life distribution F_1 and F_2 is

$$\bar{F}(w) = \int_0^\infty \bar{F}_1(w - z) dF_2(z), \quad \text{for all } w \geq 0,$$

for any fixed $t, x \geq 0$,

$$\begin{aligned} & \int_0^\infty \int_x^\infty \bar{F}(t + y) dy dx \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \bar{F}_1(x + t + y - u) dF_2(u) dy dx \\ &= \int_0^\infty \int_0^\infty \int_0^x \bar{F}_1(x + t + y - u) dF_2(u) dy dx \\ & \quad + \int_0^\infty \int_0^\infty \int_x^\infty \bar{F}_1(x + t + y - u) dF_2(u) dy dx \\ &= \int_0^\infty \int_0^x \left[\int_0^\infty \bar{F}_1(x + t + y - u) dy \right] dF_2(u) dx \\ & \quad + \int_0^\infty \int_0^\infty \left[\int_0^\infty \bar{F}_1(t + y - v) d_v F_2(v + x) \right] dy dx \\ & := A + B. \end{aligned}$$

Observe that

$$\begin{aligned} A &\leq \bar{F}_1(t) \int_0^\infty \int_0^x \left[\int_0^\infty \bar{F}_1(x + y - u) dy \right] dF_2(u) dx \\ &\leq \bar{F}(t) \int_0^\infty \int_0^\infty \int_0^x \bar{F}_1(x + y - u) dF_2(u) dy dx, \end{aligned}$$

where the first inequality follows from the NBUCA property and the second inequality trivially follows from the fact that $\bar{F}(t) \geq \bar{F}_i(t)$ for all t and $i = 1, 2$. The rest of the proof now is similar to that of Theorem 2.1 in Hu and Xie (2002). \square

For the proof of Theorem 2.2 and 2.4 we will need the following lemma. The proof is similar to Lemma 7.2(a) in Barlow and Proschan (1981).

Lemma A.1. *Assume that $W(x)$ is a Lebesgue–Stieltjes measure, not necessarily positive. If $h(x)$ is non-negative and increasing, and*

$$\int_0^\infty \int_t^\infty dW(x) dt \geq 0, \quad \text{for all } t \geq 0,$$

then $\int_0^\infty \int_t^\infty h(x) dW(x) dt \geq 0$.

Proof of Theorem 2.2. By (1.1), the order $X_1 \leq_{\text{icxa}} Y_1$ states that

$$\int_0^\infty \int_t^\infty [\bar{G}(u) - \bar{F}(u)] dx dt = \int_0^\infty \int_t^\infty [F(u) - G(u)] dx dt \geq 0, \quad t \geq 0.$$

Since the function

$$F^{n-1}(x) + F^{n-2}(x)G(x) + \cdots + F(x)G^{n-2}(x) + G^{n-1}(x)$$

is increasing and positive for all $x \geq 0$, it follows from Lemma A.1 that, for all $t \geq 0$,

$$\begin{aligned} & \int_0^\infty \int_t^\infty [\bar{G}_{N:N}(u) - \bar{F}_{N:N}(u)] dx dt \\ &= \int_0^\infty \int_t^\infty [F_{N:N}(u) - G_{N:N}(u)] dx dt \\ &= \int_0^\infty \int_t^\infty \sum_{n=1}^\infty p_N(n) [F^n(x) - G^n(x)] dx dt \\ &= \int_0^\infty \int_t^\infty \left[\sum_{n=1}^\infty p_N(n) \sum_{k=0}^{n-1} F^{n-k-1}(x) G^k(x) \right] [F(x) - G(x)] dx dt \geq 0, \end{aligned}$$

where p_N is the discrete density function of X_i . That is to say,

$$\max(X_1, \dots, X_N) \leq_{\text{icxa}} \max(Y_1, \dots, Y_N). \quad \square$$

Proof of Theorem 2.3. X_1 is NBUCA; thus, for all $t \geq 0$,

$$(X_i)_t \leq_{\text{icxa}} X_i, \quad i = 1, 2, \dots$$

By Theorem 2.2, we have,

$$\max\{(X_1)_t, \dots, (X_N)_t\} \leq_{\text{icxa}} \max\{X_1, \dots, X_N\}, \quad t \geq 0.$$

According to Li and Zuo (2004), it holds that, for any random number N ,

$$(\max\{X_1, \dots, X_N\})_t \leq_{st} \max\{(X_1)_t, \dots, (X_N)_t\}, \quad t \geq 0.$$

From transitivity, it follows that

$$(\max\{X_1, \dots, X_N\})_t \leq_{icxa} \max\{X_1, \dots, X_N\}, \quad t \geq 0.$$

That is, $\max\{X_1, \dots, X_N\}$ is NBUCA. \square

Proof of Theorem 2.4. $\min\{X_1, \dots, X_n\}$ is NWUCA, it holds that, for any $t \geq 0$,

$$(\min\{X_1, \dots, X_n\})_t \geq_{icxa} \min\{X_1, \dots, X_n\}.$$

By the fact that

$$(\min\{X_1, \dots, X_n\})_t \stackrel{st}{=} \min\{(X_1)_t, \dots, (X_n)_t\},$$

we have, for any $t \geq 0$,

$$\min\{(X_1)_t, \dots, (X_n)_t\} \geq_{icxa} \min\{X_1, \dots, X_n\}.$$

That is, for any $t \geq 0$ and $x \geq 0$,

$$\int_0^\infty \int_x^\infty \bar{F}_t^n(y) dy dx \geq \int_0^\infty \int_x^\infty \bar{F}^n(y) dy dx,$$

equivalently,

$$\int_0^\infty \int_x^\infty [\bar{F}_t^n(y) - \bar{F}^n(y)] dy dx = \int_0^\infty \int_x^\infty [\bar{F}_t(y) - \bar{F}(y)]h(y) dy dx,$$

where

$$h(y) = \bar{F}^{n-1}(y) + \bar{F}^{n-2}(y)\bar{F}_t(y) + \dots + \bar{F}(y)\bar{F}_t^{n-2}(y) + \bar{F}_t^{n-1}(y).$$

Notice that the function $[h(y)]^{-1}$ is non-negative and increasing, it follows from Lemma A.1 that, for all $x \geq 0$ and $t \geq 0$,

$$\int_0^\infty \int_x^\infty [\bar{F}_t(y) - \bar{F}(y)] dy dx \geq 0.$$

Thus, $X_t \geq_{icxa} X$, and hence X is also of NWUCA property. \square

In the proof of the following theorem we use the variation diminishing property of the totally positive kernel $k(i, t) = e^{-\lambda t} (\lambda t)^k / k!$ (see Karlin (1968) for a discussion of this property).

Proof of Theorem 4.1. Observes the following relations, which can be easily verified:

$$\int_0^\infty \int_t^\infty \bar{H}(s) ds dt = \frac{1}{\lambda^2} \sum_{k=0}^\infty \sum_{j=k}^\infty \bar{P}_j \quad (\text{A.1})$$

and

$$\int_0^\infty \int_t^\infty \bar{H}(x+s) ds dt = \frac{1}{\lambda^2} \sum_{i=0}^\infty \sum_{k=0}^\infty \left(\sum_{j=k+i}^\infty \bar{P}_j \right) \frac{e^{-\lambda t} (\lambda t)^i}{i!}. \quad (\text{A.2})$$

By assumption, from (1.3) we have

$$\bar{P}_k \sum_{i=0}^\infty \sum_{j=i}^\infty \bar{P}_j - \sum_{i=0}^\infty \sum_{j=k+i}^\infty \bar{P}_j \geq 0.$$

Since this holds for every $i, k \in N$, it follows that for every $x \geq 0$,

$$\sum_{k=0}^\infty \left[\sum_{i=0}^\infty \sum_{j=i}^\infty \bar{P}_k \bar{P}_j - \sum_{i=0}^\infty \sum_{j=k+i}^\infty \bar{P}_j \right] \frac{e^{-\lambda t} (\lambda t)^i}{i!} \geq 0.$$

By (A.1) and (A.2) this means

$$\int_0^\infty \int_t^\infty \bar{H}(x+s) ds dt \leq \bar{H}(t) \int_0^\infty \int_t^\infty \bar{H}(s) ds dt \quad \text{for all } t \geq 0.$$

i.e., $\bar{H}(t)$ is NBUCA. \square

Proof of Theorem 4.2. The theorem is obviously true for $k = 0$. For $k \geq 1$, let

$$\int_0^\infty A(x) dx = \lim_{k \rightarrow \infty} \int_0^\infty A_k(x) dx.$$

we have

$$\begin{aligned}
 & \bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \int_0^{\infty} A_j(x) dx - \sum_{j=k}^{\infty} \bar{P}_j \int_0^{\infty} A_j(x) dx \\
 &= \bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \int_0^{\infty} [A_j(x) - A(x)] dx - \sum_{j=k}^{\infty} \bar{P}_j \int_0^{\infty} [A_j(x) - A(x)] dx \\
 &= \bar{P}_k \sum_{j=i}^{\infty} \bar{P}_j \sum_{j=i}^{\infty} \int_0^{\infty} [A_j(x) - A_{l+1}(x)] dx \\
 &\quad - \sum_{j=k}^{\infty} \bar{P}_j \sum_{l=j}^{\infty} \int_0^{\infty} [A_l(x) - A_{l+1}(x)] dx \\
 &= \bar{P}_k \sum_{l=0}^{\infty} \int_0^{\infty} [A_l(x) - A_{l+1}(x)] dx - \sum_{j=0}^l \int_0^{\infty} [A_l(x) - A_{l+1}(x)] dx \sum_{j=k}^l \bar{P}_j \\
 &= \left[\bar{P}_k \sum_{l=0}^{k-1} \int_0^{\infty} [A_l(x) - A_{l+1}(x)] dx \right] \sum_{j=k}^l \bar{P}_j \\
 &\quad - \sum_{l=k}^{\infty} \int_0^{\infty} [A_l(x) - A_{l+1}(x)] dx \left[\bar{P}_k \sum_{j=0}^l \bar{P}_j - \sum_{j=k}^l \bar{P}_j \right] \geq 0. \quad \square
 \end{aligned}$$

Proof of Theorem 5.1. From (1.2) we have, for any integer $r \geq 0$

$$2 \int_0^{\infty} x^r \int_x^{\infty} \bar{v}(u) du \leq \mu_2 \int_0^{\infty} x^r \bar{F}(x) dx. \tag{A.3}$$

But

$$\int_0^{\infty} x^r \bar{F}(x) dx = E \int_0^x x^r dx = \frac{1}{r+1} E(X^{r+1}) = \frac{\mu_{r+1}}{r+1}. \tag{A.4}$$

Next,

$$\begin{aligned}
 \int_0^{\infty} x^r \int_x^{\infty} \bar{v}(u) du &= \int_0^{\infty} \bar{v}(x) \int_0^x u^r dv dx \\
 &= \int_x^{\infty} \bar{v}(x) \frac{x^{r+1}}{r+1} dx \\
 &= \frac{1}{r+1} \int_0^{\infty} x^{r+1} \int_x^{\infty} \bar{F}(u) du dx \\
 &= \frac{1}{r+1} \int_0^{\infty} \bar{F}(x) \int_0^x u^{r+1} du dx \\
 &= \frac{1}{(r+1)(r+2)} \int_0^{\infty} x^{r+2} \bar{F}(x) dx \\
 &= \frac{1}{(r+1)(r+2)(r+3)} \mu_{r+3}.
 \end{aligned}$$

The result follows from (A.3) and (A.4). \square

Proof of Theorem 5.2. The result follows directly from the central limit theory of U -statistics cf. Lee (1989). We only need to calculate the variance. Let us evaluate the variance of $\hat{\delta}(r)$ where

$$\hat{\delta}(r) = \frac{1}{n(n-1)} \sum_{i \neq j} \sum \phi_r(X_i, X_j),$$

where

$$\phi_r(X_1, X_2) = (r+2)(r+3)X_1^2 X_2^{r+1}.$$

But

$$E[\phi_r(X_1, X_2)|X_1] = (r+2)(r+3)X_1^2 \mu_{r+1} - 2X_1^{r+3} = \phi_r^{(1)}(X_1)$$

and

$$E[\phi_r(X_2, X_1)|X_1] = (r+2)(r+3)\mu_2 X_1^{r+1} - 2\mu_{r+3} = \phi_r^{(2)}(X_1).$$

Hence $\tau^2(r)$ is given by

$$\tau^2(r) = \mu^{-2r-6} V\{\phi_r^{(1)}(X_1) + \phi_r^{(2)}(X_1)\}. \quad (\text{A.5})$$

Under H_0 , and by direct calculation, the null variance is equal to

$$\begin{aligned} \tau_0^2(r) &= (r+2)(r+3)(9r^2 + 41r + 42) \\ &\quad \times (2r+2)! - 4(4r+9)[(r+3)!]^2. \quad \square \end{aligned} \quad (\text{A.6})$$

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