Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 3, 141 - 150

A note on moment inequality for harmonic used better than aged in expectation (HUBAE) class of life distributions with hypothesis testing application

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Abstract

The harmonic used better than aged in expectation HUBAE class of life distributions is considered. A moment inequality is derived for HUBAE distributions which demonstrate that if the mean life is finite, then all moments exist. Based on this inequality, a new test statistic for testing exponentiality against HUBAE is introduced . It is shown that the proposed test is simple, has high relative efficiency for some commonly used alternatives . Critical values are tabulated for sample sizes $n = 5(1)40$. The set of real data is used as a practical application of the proposed test in the medical science.

Keywords: HUBAE, exponentiality, efficiency, moments, asymptotic normality.

1 Introduction

Let X be a nonnegative continuous random variable with distribution function $F(x)$, survival function $\overline{F} = 1-F$, at age t, we define the random residual life by X_t with survival function $\overline{F_t} = \frac{\overline{F}(t+x)}{\overline{F}(t)}$, $x, t \geq 0$ and assume that X has a finite mean $\mu = E(X) = \int_0^\infty \overline{F}(u)du$. Some properties concerning the asympototic behavior of X_t as $t \to \infty$ will be used. Bhattacharjee(1982) gave the follwoing definition.

Definition(1.1). If X is nonnegative random variable, its distribution function F is said to be finitly and positively smooth if a number $\gamma \in (0, \infty)$ exists such that:

$$\lim_{t\to\infty} \frac{\overline{F}(t+x)}{\overline{F}(t)} = e^{-x\gamma}, \tag{1.1}$$

Where γ called the asymptotic decay coefficient of X. Denoting X_e be a random variable exponentialy distributed by mean $\frac{1}{\gamma}$, the following definitions imply that X_t converges to X_e in distribution written as $X \overset{d}{-} \to X_e$. This property is useful for discription of random life times of devices of unknown age.

Difinition(1.2). The distribution F is said to be used better than age UBA if for all $x, t \geq 0$

$$\int_0^\infty \overline{F}(x+t)dx \geq \overline{F}(t)e^{-\gamma x}, \tag{1.2}$$

where γ called is the asymptotic decay of X. From definition (1.2) we have the following definition:

Definition(1.3).The distribution F is said to be harmonic used better than aged in expectation HUBAE if for all $x, t \geq 0$

$$\int_x^\infty \overline{F}(t)dt \geq \mu e^{-\gamma x}, \tag{1.3}$$

where γ is asympototic decay of X.
We observe that the inequality of (1.3) is achieved when $F(x)$ has an exponential distribution with mean μ equal to the coefficient of the asymptotic decay γ ,where the exponential distribution is the only distribution which has the lack of memory property.

Alzaid (1994) showed that UBA class of life distribution is a subclass of the used better than age in excepectation class (UBAE) and that if F is IHR (increasing hazard rate) ,then F is UBA. Similar implications between UBAE, NBUE and HNBUE were given by Di Crescenzo(1999). More recently, Willmot and Cai (2000) showed that the UBA class includes the DMRL class (decreasing mean residual life) while the UBAE includes the DVRL (decreasing variance residual life).Thus we have

$$\begin{array}{ccccccc} \text{IHR} & \subset & \text{DMRL} \subset & \text{UBA} \subset & \text{UBAE} & \subset \text{HUBAE} \\ & & & & \cup & \\ & & & & \text{DVRL} & \end{array}$$

For definitions and details of the classes IHR, NBUE and DMRL, see Barlow and Prochan (1981) and for HNBUE see Klefsjo (1983) while for DVRL see Ab-Youssef (2004).

Several authores derived moments inequalities of different families of life distributions such as IHR, IHRA, NBU, NBUE, HNBUE, HRNBU, HRNBUE,NRBU, NRBUE, UBA and UBAE , cf. Ahmad(2001, 2004), Ahmad and Mugadi (2004) and Abu-Youssef (2003, 2004). Testing exponentiality against the classes of life distribution has seen a good deal of attension. For testing against IHR, we refer to Barlow and Proschan(1981) and Ahmad (1994), among others. While testing against DMRL see Ahmad(1992) and testing against DVRl see Abu-yossef (2004). Finally testing against UBA and UBAE see Ahmad (2004).

The thread that connects most work mentioned here is that a measure of departure from H_0, which is often some weighted function of F, is developed which is strictly positive under H_1 and is zero under H_0. Then, a sample version of this measure is used as test statistics and its properties are studied. In the present work, the moment inequality developed in section 2 can be used to construct test statistic for HUBAE. In section 3 this test statistic is based on sample moments of aging distribution. This test statistic is simple to drive, and has exponentially high efficiencies for the well known alternatives relative to other tests. Montecarlo null distribution critical points obtained for sample sizes 5(1)40. Finaly we apply the proposed test to real practical data in medical science given in Aboummah et al. (1994).

2 Moment Inequality

We state and prove the following theorm.
Theorem 2.1. If F is HUBAE, then

$$\gamma^{r+1}\mu_{(r+2)} \geq (r+2)!\mu, \quad r \geq 0. \tag{2.1}$$

where

$$\mu_{(r+2)} = (r+2)\int_0^\infty x^{r+1}\overline{F}(x)dx.$$

Proof. Since F is HUBAE, then

$$\nu(x) \geq \mu e^{-x\gamma}, \tag{2.2}$$

where

$$\nu(x) = \int_x^\infty \overline{F}(t)dt.$$

Multiplying both sides by x^r, $r \geq 0$, and integrating over $(0,\infty)$, w.r.t. x , then

$$\int_0^\infty x^r \nu(x)dx \geq \mu \int_0^\infty x^r e^{-x\gamma}dx \tag{2.3}$$

the left hand side of (2.3) is

$$\int_0^\infty x^r \nu(x)dx = -\frac{1}{r+1}\int_0^\infty x^{r+1}\nu^{'}(x)dx. \tag{2.4}$$

Since $\nu^{'}(x) = -\overline{F}(x)$, then

$$\int_0^\infty x^r \nu(x)dx = \frac{1}{r+1}\int_0^\infty x^{r+1}\overline{F}(x)dx = \frac{\mu_{r+2}}{(r+2)(r+1)}. \tag{2.5}$$

The right hand side of (2.3) is given by

$$\int_0^\infty \mu e^{-x\gamma}x^r dx = \frac{\mu r!}{\gamma^{r+1}} \tag{2.6}$$

By using (2.5) and (2.6), (2.1) is obtained.

3 Applications to hypotheses testing

3.1 Testing against HUBAE alternatives

Let $X_1, X_2, \ldots, X_n$ represent a random sample from a population with distribution F. We wish to test the null hypothesis $H_0 : \overline{F}$ is exponential with mean μ against $H_1 : \overline{F}$ is HUBAE and not exponential. Using theorem (2.1), we may use the following $\delta_h$as a measure of departure from H_0 infavor of H_1:

$$\delta_h = \gamma^{r+1}\mu_{(r+2)} - (r+2)!\mu \tag{3.1}$$

Note that under $H_0 : \delta_h = 0$, while under $H_1 : \delta_h > 0$. Thus to estimate δ_h by $\hat{\delta}_{h_n}$, let $X_1, X_2, \ldots, X_n$ be a random sample from F, $\hat{\gamma} = \frac{n}{\sum X_i}$ is the estimate of γ and μ is estimated by $\overline{X}$, where $\overline{X} = \frac{1}{n}\sum X_i$ is the usual sample mean . Then $\hat{\delta}_{h_n}$ is given by using (3.1) as

$$\hat{\delta}_{h_n} = \frac{1}{n}\sum_i \left\{\hat{\gamma}^{r+1}X_i^{r+2} - (r+2)!X_i\right\}. \tag{3.2}$$

to make the test statistic scale invariant, we use

$$\Delta_{h_n} = \frac{\hat{\delta}_{h_n}}{\mu}.$$

which is estimated by

$$\hat{\Delta}_{h_n} = \frac{\hat{\delta}_{h_n}}{\overline{X}}. \tag{3.3}$$

Setting $\phi(X_1) = \gamma^{r+1}X_1^{r+2} - (r+2)!X_1$, then $\hat{\Delta}_{h_n}$ in (3.3) is a U-statistic, cf. Lee (1990). The following theorem summarizes the large sample properties of $\hat{\Delta}_{h_n}$.

Theorem 3.1. As $n \to \infty$, $\sqrt{n}(\hat{\Delta}_{h_n} - \Delta_h)$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = var[\gamma^{r+1}X_1^{r+2} - (r+2)!X_1] \tag{3.4}$$

Under $H_0 : \Delta_h = 0$ and variance σ_0^2 is given by

$$\sigma_0^2 = (2r+4)! + 2((r+2)!)^2X^2 - 2(r+2)!(r+3)! \tag{3.5}$$

Proof: Since $\hat{\Delta}_{h_n}$ and $\frac{\hat{\delta}_{h_n}}{\mu}$ have the same limiting distribution, we use $\sqrt{n}(\hat{\delta}_{h_n} - \delta_{h_n})$. Now this is asymptotically normal with mean 0 and variance $\sigma^2 = var[\gamma^{r+1}\phi(X_1)]$, where

$$\phi(X_1) = X_1^{r+2} - (r+2)!X_1.$$

Then (3.4) follows.

Under H_0 $\Delta_h = E(\phi(X_1)) = 0$ and

$$\sigma_0^2 = E[X_1^{r+2} - (r+2)!X]^2. \tag{3.6}$$

Hence (3.5) follows. The Theorem is proved.

When $r = 0$,

$$\delta_h = \mu_{(2)}\gamma - 2\mu, \tag{3.7}$$

in this case $\sigma_0^2 = 8$ and the test statistic

$$\hat{\delta}_{h_n} = \frac{1}{n} \sum_i \left\{ X_i^2 \hat{\gamma} - 2X_i \right\} \tag{3.8}$$

and

$$\hat{\Delta}_{h_n} = \frac{\hat{\delta}_{h_n}}{\overline{X}^2}, \tag{3.9}$$

which is quite simple statistics.

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{h_n}/\sigma_0$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$. To illustrate the test, we calculate, via Monte Carlo Method, the empirical critical points of $\hat{\Delta}_{hn}$ in (3.9) for sample sizes 5(1)40. Tables (3.1) gives the upper percentile points for 95%, 98%, 99% . The calculations are based on 5000 simulated samples sizes $n = 5(1)40$.

Table (3.1) Critical Values of $\hat{\Delta}_{h_n}$ in(3.9)

n	95%	98%	99%
5	0.5486	0.9004	1.1503
6	0.5779	1.0236	1.2910
7	0.5940	0.9426	1.2027
8	0.6425	1.0617	1.3685
9	0.6168	1.0118	1.2873
10	0.6263	1.0513	1.3720
11	0.6315	1.0132	1.3312
12	0.6598	1.0639	1.3395
13	0.6350	0.9947	1.2981
14	0.5991	0.9572	1.4156
15	0.6061	1.0030	1.2714
16	0.6469	1.0001	1.2482
17	0.6013	0.9462	1.2626
18	0.6308	0.9413	1.2048
19	0.5709	0.9030	1.0829
20	0.5989	0.9197	1.1594
21	0.5475	0.8552	1.1012
22	0.5837	0.8623	1.0519
23	0.5838	0.8949	1.1339
24	0.5638	0.8391	1.0430
25	0.5666	0.7838	1.9764
26	0.5393	0.8062	1.0541
27	0.5472	0.8482	1.1541
28	0.5315	0.8060	0.9822
29	0.5252	0.8342	0.9951
30	0.5370	0.7905	0.9528
31	0.5101	0.7572	1.0231
32	0.5068	0.7516	0.9644
33	0.5087	0.7564	0.9648
34	0.5083	0.6984	0.8729
35	0.5216	0.7728	0.9945
36	0.4832	0.7416	0.9544
37	0.5061	0.7201	0.8976
38	0.4809	0.7142	0.8965
39	0.4758	0.7309	0.8848
40	0.4847	0.6692	0.8451

To asses how good this procedure is relative to others in the literatures, we use the concept of Pitman's asymptotic efficiency (PAE). To do this we need to evalute PAE of the proposed test and compare it with other tests. Since the above test statistic $\hat{\Delta}_{h_n}$ in (3.3) is new and no other tests are known for these class HUBAE. We may compare it with smaller classes such as (DMRL), and UBAE . Here we choose the tests K^* and $\hat{\delta}_2$ were presented by Hollander and Prochan (1975) and Ahmad (2004) respectively for decreasing mean residual life class (DMRL) and used better than aged in expectation (UBAE) class. Note that PAE of $\hat{\Delta}_{h_n}$ is given by

$$PAE(\Delta_h(\theta)) = \left\{ \frac{d}{d\theta} \Delta_E(\theta)|_{\theta \to \theta_0} \right\} / \sigma_0. \tag{3.10}$$

Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i)	Linear failure rate family	: $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$,	$x > 0, \theta > 0$
(ii)	Makeham family	: $\bar{F}_{2\theta} = e^{-x-\theta(x+e^{-x}-1)}$,	$x > 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makham families. The PAE's of these alternatives of our procedure are, respectively:

$$PAE(\Delta_h, LFR) = -\frac{(r+2)(r+3)!}{2} + (r+2)!, \qquad r \geq 0 \tag{3.11}$$

$$PAE(\Delta_h, Makeham) = -(r+2)(r+1)![r+1+\frac{1}{2^{r+2}}] + \frac{(r+2)!}{2} \tag{3.12}$$

Direct calculations of PAE of K^*, $\hat{\Delta}_2$ and $\hat{\Delta}_{h_n}$ are summarized in Table (3.2).

Table (3.2)

Distribution	K^*	$\hat{\delta}_2$	$\hat{\Delta}_{hu_n}$
F_1 Linear failure rate	0.806	0.630	1.41
F_2 Makeham	0.289	0.385	0.53

From Table (3.2), the test statistic $\hat{\Delta}_{h_n}$ is more efficient than $\hat{\Delta}_2$ and K^* for linear failure rate family, and Makeham family. **Note that:** Since $\hat{\Delta}_{h_n}$ defines a class (with parameter) r of test statistic, we choose r that the maximizes the PAE of that alternatives. If we take $r = 1$ then our test will have more efficiency than others.

4 Numerical Examples

Consider the data in Abouammoh et al (1994). These data represent 40 patients suffering from blood cancer from one of the Ministry of Health Hospital in Saudi Arabia and the ordered life times (in day are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852.

Using equation (3.9), the value of test statistics, based on the above data is $\hat{\Delta}_{h_n} = -0.0072$. This value leads to the acceptance of H_0 at the signficance level $\alpha = 0.95$ see Table (3.1). Therefore the data has't HUBAE Property.

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Received: October 7, 2005