Generalized Proportional Reversed Hazards Model

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Abstract

In this paper, we propose to generalized proportional reversed hazards model by $T_s(X^*,t) = [T_s(X,t)]^{\alpha}$, where $T_s(X,t)$ is baseline distribution function and α is a positive real number. The monotonicity of the baseline failure rates in relation to the monotonicity of the baseline hazard are studied in a general way. A set of sufficient conditions are provided for X^* to be s_IFR $[s\_DFR]$ when X is s_IFR $[s\_DFR]$. We also prove similar preservation results for the s_NBU $[s\_NWU]$ aging properties. Finally, some generalized stochastic comparisons are given.

Keywords: Generalized proportional reversed hazards model, monotonic failure rates, generalized aging classes, generalized stochastic orders

1 Introduction and preliminaries

The proportional reversed hazard model consists in describing random failure times by a relation between distribution function: $F_{X^*}(x) = [F_X(x)]^{\alpha}$,with $x \in R$ and $\alpha > 0$. One problem of interest is to determine if certain ageing properties are preserved under the transformation $X \to X^*$. *Gupta et al.* (1998) proved that if X is IFR $[DFR]$ (increasing [decreasing] failure rate) and $\alpha > 1$ $[< 1]$, then X^* is IFR $[DFR]$ ageing properties. Moreover, *Di Crescenzo* (2000) proved similar preservation results for the NBU $[NWU]$ (new better [worse] than used) and ILR $[DLR]$ (increasing [decreasing] likelihood ratio) ageing properties. For various definitions and properties of the above ageing classes, one may refer to *Bryson* and *Siddiqui* (1969), *Barlow* and *Proschan* (1981), *Deshpand et al.* (1986) and the references there in.

For any non-negative absolutely continuous random variables X with density function $f(x)$, survival function $\overline{F}(x)$ and mean μ_F, write

$$\overline{T}_0(X,t) = f(t),$$

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and

$$T_s(X,t) = \frac{\int_0^x \overline{T}_{s-1}(X,t)dt}{\mu_{s-1}(X)}, \quad s = 1, 2, ...,$$

where

$$\mu_s(X) = \int_0^{\infty} \overline{T}_s(X,t)dt, \quad s = 0, 1, 2, ... \; .$$

Note that $T_2(X,t)$ the distribution function of the first derived distribution (also called equilibrium distribution) of X, which plays an important role in the ageing concepts ($cf.$ *Deshpand et al.*, 1986). We call the first derived distribution of the first derived distribution as second derived distribution of F and so on. Thus, $T_s(X,t)$ is the distribution function of the $(s-1)^{th}$ derived distribution of F. Also, define

$$r_s(X,t) = \frac{\overline{T}_{s-1}(X,t)}{\int_x^{\infty} \overline{T}_{s-1}(X,t)dt},$$

which represents the failure rate function corresponding to $T_s(X,t)$. The function $r_s(X,t)$ is, in fact, the failure rate function of the $(s-1)^{th}$ derived distribution of F. For $s=1$, $r_1(X,t)$ is the failure rate function of X, defined as the ratio of the density to the survival function, where, for $s=2$, $r_2(X,t)$ is the reciprocal of the mean residual life function. According to this notation, in this paper, we propose to generalized proportional reversed hazards model by:

$$T_s(X^*,t) = [T_s(X,t)]^{\alpha}, \quad \text{with} \quad x \in R \quad \text{and} \quad \alpha > 0, \tag{1.1}$$

so that the reversed hazard rate function of X and X^* are proportional in a general way. For $s=1$, this model called, in literature, as *Lehman alternative*, when α is in fact a positive integer. The proposed model is flexible enough to accommodate both monotonic as well as non-monotonic failure rates even though the baseline failure rate is monotonic in general way. (For more details, please see, *Gupta et al.* (1998), *Mudholkar* and *Srivastava* (1993), *Lehman* (1953) and *Di Crescenzo* (2000)).

On the other hand, in the literature, several ageing properties in a general way depending on the generalized orderings (s_FR, s_ST *etc.*) have been defined. Let us use $\overline{U}_s(Y,t)$, $U_s(Y,t)$, $r_s(Y,t)$, and $\mu_s(Y)$ corresponding to $\overline{T}_s(X,t)$, $T_s(X,t)$, $r_s(X,t)$ and $\mu(X)$ for the variable Y. The following definitions are due to *Fagiuoli* and *Pellerey* (1993).

Definition 1.1.

A non-negative random variable X is said to be smaller than Y in:

(i) s_FR ordering (denoted by $X \leq_{s-FR} Y$) if, and only if,

$$\frac{\overline{U}_s(Y,t)}{\overline{T}_s(X,t)} \quad \text{is increasing in } t \geq 0;$$

(ii) s_ST ordering (denoted by $X \leq_{s-ST} Y$) if, and only if,

$$T_s(X,t) \geq U_s(Y,t), \quad \text{for all } t \geq 0. \tag{1.2}$$

Definition 1.2.

A non-negative random variable X is said to be:

(i) s_IFR (s_DFR) if

$$\frac{\overline{T}_{s-1}(X,t)}{\int_x^\infty \overline{T}_{s-1}(X,t)dt}, \text{ is increasing (decreasing) in } t > 0;$$

(ii) s_NBU (s_NWU) if

$$\overline{T}_s(X, x+t) \leq (\geq) \overline{T}_s(X,x)\ \overline{T}_s(X,t) \quad \text{for all } x, t \geq 0.$$

It has been observed that the following equivalences hold (see *Fagiuoli* and *Pellerey* (1993), *Nanda* (2000) and *Hu et al.* (2003)):

$0-FR \Leftrightarrow LR$; $\quad 1-FR \Leftrightarrow HR$; $\quad 2-FR \Leftrightarrow MRL$; $\quad 3-FR \Leftrightarrow VRL$;
$0-ST \Leftrightarrow WLR$; $\quad 1-ST \Leftrightarrow ST$; $\quad 2-ST \Leftrightarrow HMRL$;

$0-IFR \Leftrightarrow ILR$; $\quad 1-IFR \Leftrightarrow IFR$; $\quad 2-IFR \Leftrightarrow DMRL$;
$3-IFR \Leftrightarrow DVRL$; $\quad 1-NBU \Leftrightarrow NBU$.

One may refer to *Launer* (1984) for the definition of $DMRL$ (decreasing mean residual lives) and $DVRL$ (decreasing variance residual lives) random variables. The WLR (weak likelihood ratio), VRL (variance residual lives) and $HMRL$ (harmonic mean residual lives) orders can be found in *Singh* (1989) and *Shaked* and *Shanthikumar* (1994). For extensive review of stochastic orders and ageing classes, one may refer to *Shaked* and *Shanthikumar* (1994) and *Barlow* and *Proschan* (1981), respectively.

In the present paper, in *Section* 2 the monotonicity of the baseline failure rates in relation to the monotonicity of the baseline hazard are studied in a general way. A set of sufficient conditions are provided for X^* to be s_IFR (s_DFR) when X is s_IFR (s_DFR). In that *section*, we also prove the following preservation results of generalized ageing properties:

(i) if X is s_NBU and $\alpha > 1$, then X^* is s_NBU;

(ii) if X is s_NWU and $\alpha < 1$, then X^* is s_NWU.

Some generalized stochastic comparisons are given in *Section* 3. Indeed, we show suitable conditions such that αX and X^* are ordering according to $\leq_{s-ST}$-order. We also face the problem of preservation of some stochastic orders under the transformation $X \longrightarrow X^*$.

Throughout the paper we will use the term *increasing* in place of *non-decreasing*, and *decreasing* in place of *non-increasing*. All integrals and expectations are implicitly assumed to exist whenever they are written.

2 The proposed model and its monotonic properties

Let X be a non-negative random variable denoting the life length of a component having distribution function $T_s(X,t)$ with $T_s(X,0)=0$ and the *pdf* $\overline{T}_0(X,t)$. Let X^* be a non-negative random variable such that its function $T_s(X^*,t)$ such that is an exponentiation function of $T_s(X,t)$ i.e.

$$T_s(X^*,t) = [T_s(X,t)]^{\alpha}, \quad t>0, \quad \alpha>0.$$

Therefore, the *pdf* of X^* is

$$\overline{T}_0(X^*,t) = \alpha\,[T_s(X,t)]^{\alpha-1}\,\overline{T}_0(X,t).$$

The failure rate of X^* is given by

$$\begin{aligned} r_s(X^*,t) &= \frac{\overline{T}_0(X^*,t)}{\overline{\overline{T}}_s(X^*,t)} = \frac{\alpha\,[T_s(X,t)]^{\alpha-1}\,\overline{T}_0(X,t)}{1-[T_s(X,t)]^{\alpha}} \\ &= \alpha\; r_s(X,t)\; g_s(X,t). \end{aligned} \tag{2.2}$$

where

$$g_s(X,t) = \frac{\left[[T_s(X,t)]^{\alpha-1} - [T_s(X,t)]^{\alpha}\right]}{1-[T_s(X,t)]^{\alpha}}$$

In order to examine the monotonic properties of $r_s(X^*,t)$, we look at

$$\frac{d}{dt}\,[g_s(X,t)] = \frac{\overline{T}_0(X,x)\,[T_s(X,t)]^{\alpha-2}}{\left[\overline{T}_s(X^*,t)\right]^2}\;\left[\alpha-1-\alpha T_s(X,t)+[T_s(X,t)]^{\alpha}\right]$$

It can be verified that $g(X,t)$ is increasing (decreasing) function of t for $\alpha>1$ ($\alpha<1$). Thus we have the following results:

Theorem 2.1.

(i) If $\alpha>1$ and $X \in s\_IFR$, then $X^* \in s\_IFR$.

(ii) If $\alpha<1$ and $X \in s\_DFR$, then $X^* \in s\_DFR$.

Corollary 2.1. (*Gupta et al*, 1998)

A random variable X is

(i) If $\alpha>1$ and $X \in IFR$, then $X^* \in IFR$.

(ii) If $\alpha<1$ and $X \in DFR$, then $X^* \in DFR$.

Proof.

Taking $s=1$ in (i) and (ii) of the above theorem, the result follows.

In order to obtain more general condition, we define the following classes of failure rates for a general $r_s(X,t)$:

(*i*) If $\frac{d}{dt}r_s(X,t) > 0$ for all t, we say that $X \in s\_IFR$;

(*ii*) If $\frac{d}{dt}r_s(X,t) < 0$ for all t, we say that $X \in s\_DFR$.

To determine the nature of the failure rate, define

$$\phi(X,t) = -\frac{\frac{d}{dt}\left[\overline{T}_0(X,t)\right]}{\overline{T}_0(X,t)},$$

where $\overline{T}_0(X,t)$ is the *pdf* corresponding to $r_s(X,t)$. The following result gives a characterizations of the generalized ageing classes.

Lemma 2.1.

(*a*) If $\frac{d}{dt}\left[\phi(X,t)\right] > 0$ for all $t > 0$, then $X \in s\_IFR$;

(*b*) If $\frac{d}{dt}\left[\phi(X,t)\right] < 0$ for all $t > 0$, then $X \in s\_DFR$.

We now apply the above Lemma to $T_s(X^*,t) = \left[T_s(X,t)\right]^{\alpha}$, we have

$$\frac{d}{dt}\left[\overline{T}_0(X^*,t)\right] = \alpha(\alpha-1)\left[T_s(X,t)\right]^{\alpha-2}\left[\overline{T}_0(X,t)\right]^2 + \alpha\left[T_s(X,t)\right]^{\alpha-1}\frac{d}{dt}\left[\overline{T}_0(X,t)\right].$$

This gives

$$\frac{d}{dt}\left[\phi(X^*,t)\right] = (1-\alpha)\frac{d^2}{dt^2}\left[\ln T_s(X,t)\right] - \frac{d^2}{dt^2}\left[\ln \overline{T}_0(X,t)\right].$$

Thus, $X^* \in s\_IFR$, if $\alpha > 1$ and

$$\frac{d^2}{dt^2}\left[\ln T_s(X,t)\right] < \frac{1}{(1-\alpha)}\frac{d^2}{dt^2}\left[\ln \overline{T}_0(X,t)\right], \quad \text{for all } t > 0.$$

Also, $X^* \in s\_IFR$, if $\alpha < 1$ and

$$\frac{d^2}{dt^2}\left[\ln T_s(X,t)\right] > \frac{1}{(1-\alpha)}\frac{d^2}{dt^2}\left[\ln \overline{T}_0(X,t)\right], \quad \text{for all } t > 0.$$

Similar conditions can be obtained for X^* to belong to s_DFR.

Next we give the second preservation results of the generalized ageing classes under the generalized proportional reversed hazard model. Before to give the results we need to give the following lemma (see B.3.a of *Chapter* 16 of *Marshall* and *Olkin*, 1979).

Lemma 2.2.

Let ψ be a real function defined on an interval $I \subset R$, and let $x_1 < y_1 \leq y_2$ and $x_1 \leq x_2 < y_2$;

(i) if ψ is convex on I, then

$$\frac{\psi(y_1)-\psi(_1)}{y_1-x_1} \leq \frac{\psi(y_2)-\psi(x_2)}{y_2-x_2} \tag{2.3}$$

(ii) if ψ is concave on I, then the inequality in (2.3) is reversed.

Theorem 2.2.

The following statements hold:

(i) if X is s_NBU and $\alpha > 1$, then X^* is s_NBU;

(ii) if X is s_NWU and $\alpha < 1$, then X^* is s_NWU.

Proof.

Let $x_1 = T_s(X,x)T_s(X,t)$, $x_2 = T_s(X,x)$, $y_1 = T_s(X,t)$, $y_2 = T_s(X,x+t)$, for $x,t \geq 0$. We set $\psi(z) = z^\alpha$, with $\alpha > 1$, so that ψ is convex on $[0,\infty)$. Makin use of *Eq.* (2.3) one has

$$[T_s(X,x+t)-T_s(X,x)]\left\{[T_s(X,t)]^\alpha-[T_s(X,x)T_s(X,t)]^\alpha\right\}$$

$$\leq [T_s(X,t)-T_s(X,x)]\left\{[T_s(X,x+t)]^\alpha-[T_s(X,x)]^\alpha\right\} \tag{2.4}$$

Moreover, if X is s_NBU, for all $x,t \geq 0$ it follows that

$$T_s(X,x+t)-T_s(X,x) \geq T_s(X,t)-T_s(X,x)T_s(X,t).$$

Hence, in order that (2.4) holds it must be

$$[T_s(X,t)]^\alpha-[T_s(X,x)T_s(X,t)]^\alpha \leq [T_s(X,x+t)]^\alpha-[T_s(X,x)]^\alpha,$$

i.e.,

$$1-[T_s(X,x+t)]^\alpha \leq \left\{1-[T_s(X,x)]^\alpha\right\}\left\{1-[T_s(X,t)]^\alpha\right\}.$$

The latter inequality, by virtue of (1.1), becomes

$$\overline{T}_s(X^*,x+t) \leq \overline{T}_s(X^*,t)\,\overline{T}_s(X^*,x), \qquad \text{for all } x,t \geq 0.$$

Thus, X^* is s_NBU, so that statement (i) holds.

Being z^α concave on $[0,\infty)$ when $\alpha < 1$, making use of (ii) of *Lemma* 2.2 one can prove statement (ii) similarly.

Corollary 3.2. (*Di Crescenzo,* 2000)

The following statements hold:

(i) if X is NBU and $\alpha > 1$, then X^* is NBU;

(ii) if X is NWU and $\alpha < 1$, then X^* is NWU.

3 Some properties based on generalized stochastic orders

In this section we show some of the generalized proportional reversed hazard model which are based on generalized stochastic comparisons. To this purpose, let us recall that given two non-negative random variables X and Y, X is said to be smaller than Y in the generalized hazard rate order if

$$r_s(X,t) \geq r_s(Y,t), \qquad \text{for all } t.$$

According to (1.1) we conclude that, if $\alpha > 1$, $r_s(X^*,t) \leq r_s(X,t)$ for all $t > 0$ and if $\alpha < 1$, $r_s(X^*,t) \geq r_s(X,t)$ for all $t > 0$. This can also be seen by considering X^* as a weighted version of X with weight function $w_s(X^*,t) = \alpha\left[T_s(X,t)\right]^{\alpha-1}$. Since $\alpha > 1$ ($\alpha < 1$) implies that $w_s(X^*,t)$ is increasing (decreasing) function of t, we have $r_s(X^*,t) \leq r_s(X,t)$ if $\alpha > 1$ and $r_s(X^*,t) \geq r_s(X,t)$ if $\alpha < 1$, for all $t > 0$.

Being

$$T_s(\alpha X,t) = T_s(X,\frac{t}{\alpha}), \qquad t \in \mathbf{R}, \tag{3.1}$$

the following result holds: if $\alpha < 1$ $[\alpha > 1]$ then $X \geq_{s-ST} \alpha X$ $[X \leq_{s-ST} \alpha X]$. Thus, X^* and αX are both larger [smaller] than X when $\alpha < 1$ $[\alpha > 1]$. Next we give some generalized stochastic comparisons results for αX and X^*, where α is in common for such random variables. Before to give the results we need to give the notion of star-shaped function (see *Marshall* and *Olkin*, 1979).

Recall that, a function $g : [0,\infty) \longrightarrow [0,\infty)$ is said to be starshaped [antistarshaped] if $g(0) = 0$ and $g(t)/t$ is increasing [decreasing] in $t \geq 0$, or equivalently if $\varphi(\alpha t) \geq [\leq] \alpha\varphi(t)$, for all $t \geq 0$ and $0 \leq \alpha \leq 1$.

Let us introduce the generalized cumulative reversed hazard rate function of X, defined by From (3.1) one has

$$U_s(X,t) = -\ln T_s(X,t),$$

or equivalently,

$$T_s(X,t) = \exp\left\{-U_s(X,t\right\}, \qquad t > 0. \tag{3.2}$$

Theorem 3.1.

Let $\frac{1}{U_s(X,t)}$ be starshaped [antistarshaped]:

(i) if $\alpha < 1$ then $\alpha X \leq_{s-ST} X^*$ $[\alpha X \geq_{s-ST} X^*]$;

(ii) if $\alpha > 1$ then $\alpha X \geq_{s-ST} X^*$ $[\alpha X \leq_{s-ST} X^*]$.

Proof.

If $\frac{1}{U_s(X,t)}$ is starshaped and $\alpha < 1$ one has

$$U_s(X,t) \leq \alpha U_s(X, \alpha t),$$

so that due to (3.2) it follows that

$$T_s(X,t) = \exp\{-U_s(X,t\} \geq \exp\{-\alpha U_s(X,\alpha t\} = [T_s(X,\alpha t)]^{\alpha}, \quad \text{for all } t > 0.$$

Hence, from (1.1) and (3.1) we have

$$T_s(\alpha X, t) = T_s(X, \frac{t}{\alpha}) \geq [T_s(X,t)]^{\alpha} = T_s(X^*, t),$$

Appealing to (1.2), the result follow.

Suppose now that $\frac{1}{U_s(X,t)}$ is starshaped and $\alpha > 1$ one then has

$$\begin{aligned} T_s(X^*, t) &= [T_s(X,t]^{\alpha} \\ &= \exp\{-\alpha U_s(X,t)\} \\ &\geq \exp\left\{-\alpha U_s(X, \frac{t}{\alpha})\right\} \\ &= T_s(X, \frac{t}{\alpha}) = T_s(\alpha X, t), \quad \text{for all } t > 0, \end{aligned}$$

so that $\alpha X \geq_{s-ST} X^*$. When $\frac{1}{U_s(X,t)}$ is antistarshaped the proof is similar.

We conclude this section by facing the problem of preservation of some generalized stochastic orders under the transformation $X \longrightarrow X^*$. For two non-negative absolutely continuous random variables X and Y, denoted by X and Y the corresponding random variables having generalized proportional reversed hazard, where is in common for these variables. Hence, recalling (2.2) we have

$$r_s(Y,t) = \alpha \; r_s(Y,t) \; g_s(Y,t), \tag{3.3}$$

where

$$g_s(Y,t) = \frac{\left[[T_s(Y,t)]^{\alpha-1} - [T_s(Y,t)]^{\alpha}\right]}{1 - [T_s(Y,t)]^{\alpha}}.$$

Theorem 3.2.

The following statements hold:

(i) $X \leq_{s-ST} Y$ if and only if $X^* \leq_{s-ST} Y^*$;

(ii) $X \leq_{s-FR} Y$ if and only if $X^* \leq_{s-FR} Y^*$.

Proof.

The proof of (i) easily follow from (1.1). In order to prove (ii) we note that expression (2.2) and (3.3) imply

$$r_s(X^*,t) - r_s(Y^*,t) = \alpha \left[r_s(X,t) g_s(X,t) - r_s(Y,t) g_s(Y,t) \right].$$

From assumption $X \leq_{s-FR} Y$ one has $r_s(X,t) \geq r_s(Y,t)$ for all $t < a$. Moreover, $T_s(X,t) \geq T_s(Y,t)$ for all $t \in R$. Hence, as function

$$\lambda(k) = \frac{k^{\alpha-1} - k^{\alpha}}{1 - k^{\alpha}},$$

is increasing in $k \in (0,1)$ when $\alpha > 1$, we have

$$g_s(X,t) - g_s(Y,t) = \lambda \left[T_s(X,t) \right] - \lambda \left[T_s(Y,t) \right] \geq 0 \quad \text{for all } t < k.$$

It then follows that $r_s(X^*,t) \geq r_s(Y^*,t)$.

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References

Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing.* To Begin with. Silver Spring, M D.

Bryson, M. C. and Siddiqui, M. M. (1969). Some criteria for ageing. *Journal of the American Statistical Association,* **64,** 1472-1483.

Di Crescenzo*, A. (*2000). Some results on the proportional reversed hazard model. *Statistics* & *Prob*ability Letters, **50**, 313-321.

Deshpande, J. V., Kochar, S. C. and Singh, H. (1986). Aspects of positive ageing. *Journal of Applied Probability*, **23**, 748-758.

Fagiuoli, E. and Pellerey, F. (1993). New partial orderings and applications. *Naval Research Logistics*, **40**, 829-842.

Gupta, R. C., Gupta, R. D., and Gupta, P. L. (1998). Modeling failure time data by Lehman alternatives. *Communication in Statistics, Theory* & *Methods*, **27**, 887-904.

Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association*, **83**, 414-425.

Hu, T., Ma, M. and Nanda, A. (2003). Some results about generalized ageing classes of life distributions. *Technical report,* Department of Statistics and Finance, University of Science and Technology of China.

Lehman, E. L., (1953). The power of rank tests. *Annals of Mathematical Statistics*, **24**, 28-43.

Launer, R. L. (1984). Inequalities for *NBUE* and *NWUE* life distributions. *Operations Research*, **32**, 660-667.

Marshll, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and its Applications*. Academic Press, New York.

Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bath tub failure data. *IEEE Transactions on Reliability*, **42**, 299-302.

Nanda, A. (2000). Generalized ageing classes in terms of Laplace transforms. *Snakhya A*, 258-266.

Sing, H. (1989). On partial orderings of life distributions. *Naval Research Logistics*, 103-110.

Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic Orders and Their Applications*. Academic Press, New York.